L^p-Square Function Estimates on Spaces of Homogeneous Type and on Uniformly Rectifiable Sets

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Abstract

We establish square function estimates for integral operators on uniformly rectifiable sets by proving a local $T(b)$ theorem and applying it to show that such estimates are stable under the so-called big pieces functor. More generally, we consider integral operators associated with Ahlfors-David regular sets of arbitrary codimension in ambient quasi-metric spaces. The local $T(b)$ theorem is then used to establish an inductive scheme in which square function estimates on so-called big pieces of an Ahlfors-David regular set are proved to be sufficient for square function estimates to hold on the entire set. Extrapolation results for L^p and Hardy space versions of these estimates are also established. Moreover, we prove square function estimates for integral operators associated with variable coefficient kernels, including the Schwartz kernels of pseudodifferential operators acting between vector bundles on subdomains with uniformly rectifiable boundaries on manifolds.

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Contents

1 Introduction

The purpose of this work is three-fold: first, to develop the so-called "local $T(b)$ theory" for square functions in a very general context, in which we allow the ambient space to be of homogeneous type, and in which the "boundary" of the domain is of arbitrary (positive integer) co-dimension; second, to use a special case of this local $T(b)$ theory to establish boundedness, for a rather general class of square functions, on uniformly rectifiable sets of codimension one in Euclidean space; and third, to establish an extrapolation principle whereby an L^p (or even weak-type L^p) estimate for a square function, for *one* fixed p, yields a full range of L^p bounds. We shall describe these results in more detail below, but let us first recall some of the history of the development of the theory of square functions.

Referring to the role square functions play in mathematics, E. Stein wrote in 1982 (cf. [70]) that "*[square] functions are of fundamental importance in analysis, standing as they do at the crossing of three important roads many of us have travelled by: complex function theory, the Fourier transform (or orthogonality in its various guises), and real-variable methods.*" In the standard setting of the unit disc $\mathbb{D}$ in the complex plane, the classical square function Sf of some $f:\mathbb{T}\to\mathbb{C}$ (with $\mathbb{T}:=\partial\mathbb{D}$) is defined in terms of the Poisson integral $u_f(r,\omega)$ of f in $\mathbb{D}$ (written in polar coordinates) by the formula

$$(Sf)(z):=\Big(\int_{(r,\omega)\in\Gamma(x)}|(\nabla u_f)(r,\omega)|^2\,r\,dr\,d\omega\Big)^{1/2},\qquad z\in\mathbb{T},\tag{1.1}$$

where $\Gamma(z)$ stands for the Stolz domain $\{(r,\omega):\,|\arg(z)-\omega|<1-r<\frac{1}{2}\}$ in $\mathbb{D}$. Let v denote the (normalized) complex conjugate of u_f in $\mathbb{D}$. Then, if the analytic function $F:=u_f+iv$ is one-to-one, the quantity $(Sf)(z)^2$ may be naturally interpreted as the area of the region $F(\Gamma(z))\subseteq\mathbb{C}$ (recall that $\det(DF)=|\nabla u_f|^2$). The operator (1.1) was first considered by Lusin and the observation just made justifies the original name for (1.1) as Lusin's area function (or Lusin's area integral). A fundamental property of S, originally proved by complex methods (cf. [12, Theorem 3, pp. 1092-1093], and [30] for real-variable methods) is that

$$\|Sf\|_{L^p(\mathbb{T})}\approx\|f\|_{H^p(\mathbb{T})}\quad\text{ for }\ p\in(0,\infty),\tag{1.2}$$

which already contains the H^p-boundedness of the Hilbert transform. Indeed, if $F=u+iv$ is analytic then the Cauchy-Riemann equations entail $|\nabla u|=|\nabla v|$ and, hence, $S(u|_{\mathbb{T}})=S(v|_{\mathbb{T}})$. In spite of the technical, seemingly intricate nature of (1.1) and its generalizations to higher dimensions, such as

$$(Sf)(x):=\Big(\int_{|x-y|<t}|(\nabla u_f)(y,t)|^2\,t^{1-n}dydt\Big)^{1/2},\qquad x\in\mathbb{R}^n:=\partial\mathbb{R}^{n+1}_+,\tag{1.3}$$

a great deal was known by the 1960's about the information encoded into the size of Sf, measured in L^p, thanks to the pioneering work of D.L. Burkholder, A.P. Calderón, C. Fefferman, R.F. Gundy, N. Lusin, J. Marcinkiewicz, C. Segovia, M. Silverstein, E.M. Stein, and A. Zygmund, among others. See, e.g., [10], [11], [12], [30], [67], [69], [70], [71], and the references therein.

Subsequent work by B. Dahlberg, E. Fabes, D. Jerison, C. Kenig and others, starting in the late 1970's (cf. [20], [21], [28], [55], [65]), has brought to prominence the relevance of square function estimates in the context of partial differential equations in non-smooth settings, whereas work by D. Jerison and C. Kenig [53] in the 1980's as well as G. David and S. Semmes

in the 1990's (cf. [25], [26]) has lead to the realization that square function estimates are also intimately connected with the geometry of sets (especially geometric measure theoretic aspects). More recently, square function estimates have played an important role in the solution of the Kato problem in [45], [41], [3].

The operator S defined in (1.1) is obviously non-linear but the estimate

$$\|Sf\|_{L^p} \leq C\|f\|_{H^p} \tag{1.4}$$

may be linearized by introducing a suitable (linear) vector-valued operator. Specifically, set $\Gamma := \{(z,t) \in \mathbb{R}^{n+1}_+ : |z| < t\}$ and consider the Hilbert space

$$\mathscr{H} := \Big\{h : \Gamma \to \mathbb{C}^n : \ h \text{ is measurable and } \|h\|_{\mathscr{H}} := \Big(\int_\Gamma |h(z,t)|^2 t^{1-n}\,dtdz\Big)^{\frac{1}{2}} < \infty\Big\}. \tag{1.5}$$

Also, let $\widetilde{S}f : \mathbb{R}^n \to \mathscr{H}$ be defined by the formula

$$\Big((\widetilde{S}f)(x)\Big)(z,t) := (\nabla u_f)(x-z,t), \qquad \forall\, x \in \mathbb{R}^n, \ \forall\,(z,t) \in \Gamma, \tag{1.6}$$

i.e., $\widetilde{S}$ is the integral operator (mapping scalar-valued functions defined on $\mathbb{R}^n$ into $\mathscr{H}$-valued functions defined on $\mathbb{R}^n$), whose kernel $k : \mathbb{R}^n \times \mathbb{R}^n \setminus \text{diagonal} \to \mathscr{H}$, which is of convolution type, is given by $(k(x,y))(z,t) := (\nabla P_t)(x-y-z)$, for all $x, y \in \mathbb{R}^n$, $x \neq y$, and $(z,t) \in \Gamma$, where $P_t(x)$ is the Poisson kernel in $\mathbb{R}^{n+1}_+$. Then, if $L^p(\mathbb{R}^n, \mathscr{H})$ stands for the Bôchner space of $\mathscr{H}$-valued, p-th power integrable functions on $\mathbb{R}^n$, it follows that

$$\|Sf\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{H^p(\mathbb{R}^n)} \iff \|\widetilde{S}f\|_{L^p(\mathbb{R}^n,\mathscr{H})} \leq C\|f\|_{H^p(\mathbb{R}^n)}. \tag{1.7}$$

The relevance of the linearization procedure described in (1.5)-(1.7) is that it highlights the basic role of the case $p = 2$ in (1.4). This is because the operator $\widetilde{S}$ falls within the scope of theory of Hilbert space-valued singular integral operators of Calderón-Zygmund type for which boundedness on L^2 automatically extrapolates to the entire scale L^p, for $1 < p < \infty$ (the extension to the case when $p \leq 1$ makes use of other specific features of $\widetilde{S}$).

From the point of view of geometry, what makes the above reduction to the case $p = 2$ work is the fact that the upper-half space has the property that $x + \Gamma \subseteq \mathbb{R}^{n+1}_+$ for every $x \in \partial\mathbb{R}^{n+1}_+$. Such a cone property actually characterizes Lipschitz domains (cf. [47]), in which scenario this is the point of view adopted in [64, Theorem 4.11, p. 73].

Hence, S may be eminently regarded as a singular integral operator with a Hilbert space-valued Calderón-Zygmund kernel and, as such, establishing the L^2 bound

$$\|\widetilde{S}f\|_{L^2(\mathbb{R}^n,\mathscr{H})} \leq C\|f\|_{L^2(\mathbb{R}^n)} \tag{1.8}$$

is of basic importance to jump-start the study of the operator S. Now, as is well-known (and easy to check; see, e.g., [71, pp. 27-28]), (1.8) follows from Fubini's and Plancherel's theorems.

For the goals we have in mind in the present work, it is worth recalling a quote from C. Fefferman's 1974 ICM address [29] where he writes that "*When neither the Plancherel theorem nor Cotlar's lemma applies, L^2-boundedness of singular operators presents very hard problems, each of which must (so far) be dealt with on its own terms.*" For scalar singular integral operators, this situation began to be remedied in 1984 with the advent of the $T(1)$-Theorem, proved by G. David and J.-L. Journé in [23]. This was initially done in the Euclidean

setting, using Fourier analysis methods. It was subsequently generalized and refined in a number of directions, including the extension to spaces of homogeneous type by R. Coifman (unpublished, see the discussion in [13]), and the $T(b)$ Theorems proved by A. McIntosh and Y. Meyer in [58], and by G. David, J.L. Journé and S. Semmes in [24]. The latter reference also contains an extension to the class of singular-integral operators with matrix-valued kernels. The more general case of operator-valued kernels has been treated by Figiel [31] and by T. Hytönen and L. Weis [52], who prove $T(1)$ Theorems in the spirit of the original work in [23] for singular integrals associated with kernels taking values in Banach spaces satisfying the UMD property. Analogous $T(b)$ theorems were obtained by Hÿtonen [49] (in Euclidean space) and by Hÿtonen and Martikainen [50] (in a metric measure space). Yet in a different direction, initially motivated by applications to the theory of analytic capacity, L^2-boundedness criteria which are local in nature appeared in the work of M. Christ [14]. Subsequently, Christ's local $T(b)$ theorem has been extended to the setting of non-doubling spaces by F. Nazarov, S. Treil and A. Volberg in [66]. Further extensions of the local $T(b)$ theory for singular integrals appear in [5], [7], [6] and [51].

Much of the theory mentioned in the preceding paragraph has also been developed in the context of square functions, as opposed to singular integrals. In the convolution setting discussed above, (1.8) follows immediately from Plancherel's theorem, but the latter tool fails in the case when $\mathbb{R}^{n+1}_{+}$ is replaced by a domain whose geometry is rough (so that, e.g., the cone property is violated), and/or one considers a square-function operator whose integral kernel $\theta(x, y)$ is no longer of convolution type (as was the case for $\widetilde{S}$). A case in point is offered by the square-function estimate of the type

$$\int_0^\infty \|\Theta_t f\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t} \leq C\|f\|^2_{L^2(\mathbb{R}^n)}, \tag{1.9}$$

where

$$\big(\Theta_t f\big)(x) := \int_{\mathbb{R}^n} \theta_t(x, y) f(y)\, dy, \qquad x \in \mathbb{R}^n, \qquad t > 0, \tag{1.10}$$

with $\{\theta_t(\cdot,\cdot)\}_{t>0}$ a standard Littlewood-Paley family, i.e., satisfying for some exponent $\alpha > 0$,

$$|\theta_t(x, y)| \leq C\frac{t^\alpha}{(t + |x - y|)^{n+\alpha}} \quad \text{and} \tag{1.11}$$

$$|\theta_t(x, y) - \theta_t(x, y')| \leq C\frac{|y - y'|^\alpha}{(t + |x - y|)^{n+\alpha}} \ \text{ if } \ |y - y'| < t/2. \tag{1.12}$$

Then, in general, linearizing estimate (1.9) in a manner similar to (1.7) yields an integral operator which is no longer of convolution type. As such, Plancherel's theorem is no longer directly effective in dealing with (1.9) given that the task at hand is establishing the L^2-boundedness of a variable kernel (Hilbert-valued) singular integral operator. However, M. Christ and J.-L. Journé have shown in [15] (under the same size/regularity conditions in (1.11)-(1.12)) that the square function estimate (1.9) is valid if the following Carleson measure condition holds:

$$\sup_{Q\subseteq\mathbb{R}^n} \Big(\int_0^{\ell(Q)} \fint_Q |(\Theta_t 1)(x)|^2 \,\frac{dxdt}{t}\Big) < \infty, \tag{1.13}$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$. The latter result is also implicit in the work of Coifman and Meyer [17]. Moreover, S. Semmes' has shown in [68] that, in the above setting, (1.9) holds if there exists a para-accretive function b such that (1.13) holds with "1" replaced by "b".

Refinements of Semmes' global $T(b)$ theorem for square functions, in the spirit of M. Christ's local $T(b)$ theorem for singular integrals [14], have subsequently been established in [2], [39], [40]. The local $T(b)$ theorem for square functions which constitutes the main result in [40] reads as follows. Suppose Θ_t is as in (1.10) with kernel satisfying (1.11)-(1.12) as well as

$$|\theta_t(x,y) - \theta_t(x',y)| \leq C\frac{|x-x'|^\alpha}{(t+|x-y|)^{n+\alpha}} \quad \text{if} \quad |x-x'| < t/2. \tag{1.14}$$

In addition, assume that there exists a constant $C_o \in (0,\infty)$ along with an exponent $q \in (1,\infty)$ and a system $\{b_Q\}_Q$ of functions indexed by dyadic cubes Q in $\mathbb{R}^n$, such that for each dyadic cube $Q \subseteq \mathbb{R}^n$ one has:

(i) $\int_{\mathbb{R}^n} |b_Q(x)|^q\,dx \leq C_o|Q|$;

(ii) $\frac{1}{C_o}|Q| \leq \left|\int_{\mathbb{R}^n} b_Q(x)\,dx\right|$;

(iii) $\int_Q \left(\int_0^{\ell(Q)} |(\Theta_t b_Q)(x)|^2\,\frac{dt}{t}\right)^{q/2} dx \leq C_o|Q|$.

Then the square function estimate (1.9) holds. The case $q = 2$ of this theorem does not require (1.14) (just regularity in the second variable, as in (1.12))[1], and was already implicit in the solution of the Kato problem in [45], [41], [3]. It was formulated explicitly in [2], [39]. An extension of the result of [40] to the case that the half-space is replaced by $\mathbb{R}^{n+1} \setminus E$, where E is a closed Ahlfors-David regular set (cf. Definition 2.9 below) of Hausdorff dimension n, appears in [33]. The latter extension has been used to prove a result of *free boundary* type, in which higher integrability of the Poisson kernel, in the presence of certain natural background hypotheses, is shown to be equivalent to uniform rectifiability (cf. Definition 5.4 below) of the boundary [43], [44]. Further extensions of the result of [40], to the case in which the kernel θ_t and pseudo-accretive system b_Q may be matrix-valued (as in the setting of the Kato problem), and in which θ_t need no longer satisfy the pointwise size and regularity conditions (1.11)-(1.12), will appear in the forthcoming Ph.D. thesis of A. Grau de la Herran [32].

A primary motivation for us in the present work is the connection between square function bounds (or their localized versions in the form of "Carleson measure estimates"), and a quantitative, scale invariant notion of rectifiability. This subject has been developed extensively by David and Semmes [25], [26] (but with some key ideas already present in the work of P. Jones [54]). Following [25], [26], we shall give in the sequel (cf. Definition 5.4), a precise definition of the property that a closed set E is "Uniformly Rectifiable" (UR), but for now let us merely mention that UR sets are the ones on which "nice" singular integral operators are bounded on L^2. David and Semmes have shown that these sets may also be characterized via certain square function estimates, or equivalently, via Carleson measure estimates. For example, let $E \subset \mathbb{R}^{n+1}$ be a closed set of codimension one, which is (n-dimensional) Ahlfors-David regular (ADR) (cf. Definition 2.9 below). Then E is UR if and only if we have the Carleson measure estimate

$$\sup_B r^{-n} \int_B \left|\left(\nabla^2 \mathcal{S}1\right)(x)\right|^2\,\operatorname{dist}(x,E)\,dx < \infty, \tag{1.15}$$

[1] In fact, even the case $q \neq 2$ does not require (1.14), if the vertical square function is replaced by a conical one; see [32] for details.

where the supremum runs over all Euclidean balls $B := B(z,r) \subseteq \mathbb{R}^{n+1}$, with $r \leq \operatorname{diam}(E)$, and center $z \in E$, and where $\mathcal{S}f$ is the harmonic single layer potential of the function f, i.e.,

$$\mathcal{S}f(x) := c_n \int_E |x-y|^{1-n} f(y)\, d\mathcal{H}^n(y), \qquad x \in \mathbb{R}^{n+1} \setminus E. \tag{1.16}$$

Here $\mathcal{H}^n$ denotes n-dimensional Hausdorff measure. For an appropriate normalizing constant $c_n|x|^{1-n}$ is the usual fundamental solution for the Laplacian in $\mathbb{R}^{n+1}$. We refer the reader to [26] for details, but see also Section 4 where we present some related results. We note that by "$T1$" reasoning (cf. Section 3 below), (1.15) is equivalent to the square function bound

$$\int_{\mathbb{R}^{n+1}\setminus E} \big|(\nabla^2 \mathcal{S}f)(x)|^2 \operatorname{dist}(x,E)\, dx \leq C \int_E |f(x)|^2\, d\mathcal{H}^n(x)\,. \tag{1.17}$$

Using an idea of P. Jones [54], one may derive, for UR sets, a quantitative version of the fact that rectifiability may be characterized in terms of existence a.e. of approximate tangent planes. Again, a Carleson measure is used to express matters quantitatively. For $x \in E$ and $t > 0$ we set

$$\beta_2(x,t) := \inf_P \left(\frac{1}{t^n} \int_{B(x,t)\cap E} \left(\frac{\operatorname{dist}(y,P)}{t} \right)^2 d\mathcal{H}^n(y) \right)^{1/2}, \tag{1.18}$$

where the infimum runs over all n-planes P. Then a closed, ADR set E of codimension one is UR if and only if the following Carleson measure estimate holds on $E \times \mathbb{R}_+$:

$$\sup_{x_0 \in E,\, r>0} r^{-n} \int_0^r \int_{B(x_0,t)\cap E} \beta_2(x,t)^2\, d\mathcal{H}^n(x) \frac{dt}{t} \, < \, \infty. \tag{1.19}$$

See [25] for details, and for a formulation in the case of higher codimension. A related result, also obtained in [25], is that a set E as above is UR if and only if, for every odd $\psi \in C_0^\infty(\mathbb{R}^{n+1})$, one has the following discrete square function bound

$$\sum_{k=-\infty}^{\infty} \int_E \left| \int_E 2^{-kn} \psi\big(2^{-k}(x-y)\big) f(y)\, d\mathcal{H}^n(y) \right|^2 d\mathcal{H}^n(x) \leq C_\psi \int_E |f(x)|^2\, d\mathcal{H}^n(x)\,. \tag{1.20}$$

Again, there is a Carleson measure analogue, and also a version for sets E of higher codimension.

The following theorem collects some of the main results in our present work. It generalizes results described earlier in the introduction, which were valid in the codimension one case, and in which the ambient space $\mathscr{X}$ was Euclidean. To state it, recall that (in a context to be made precise below) a measurable function $b : E \to \mathbb{C}$ is called para-accretive if it is essentially bounded and there exist constants $c, C \in (0,\infty)$ such that the following conditions are satisfied:

$$\forall\, Q \in \mathbb{D}(E) \quad \exists\, \widetilde{Q} \in \mathbb{D}(E) \quad \text{such that} \quad \widetilde{Q} \subseteq Q, \quad \ell(\widetilde{Q}) \geq c\ell(Q), \quad \left| ⨍_{\widetilde{Q}} b\, d\sigma \right| \geq C. \tag{1.21}$$

Other relevant definitions will be given in the sequel.

Theorem 1.1. *Suppose that $(\mathscr{X}, \rho, \mu)$ is an m-dimensional* ADR *space for some $m > 0$ and fix a number $d \in (0, m)$. Also, let*

$$\theta : (\mathscr{X} \times \mathscr{X}) \setminus \{(x, x) : x \in \mathscr{X}\} \longrightarrow \mathbb{R} \tag{1.22}$$

be a function which is Borel measurable with respect to the product topology $\tau_\rho \times \tau_\rho$, and which has the property that there exist finite positive constants C_θ, α, υ such that for all $x, y \in \mathscr{X}$ with $x \neq y$ the following hold:

$$|\theta(x, y)| \leq \frac{C_\theta}{\rho(x, y)^{d+\upsilon}}, \tag{1.23}$$

$$|\theta(x, y) - \theta(x, \widetilde{y})| \leq C_\theta \frac{\rho(y, \widetilde{y})^\alpha}{\rho(x, y)^{d+\upsilon+\alpha}}, \quad \forall\, \widetilde{y} \in \mathscr{X} \setminus \{x\} \ \text{ with } \ \rho(y, \widetilde{y}) \leq \tfrac{1}{2}\rho(x, y). \tag{1.24}$$

Assume that E is a closed subset of $(\mathscr{X}, \tau_\rho)$ and that σ is a Borel regular measure on $(E, \tau_{\rho|_E})$ such that $\big(E, \rho\big|_E, \sigma\big)$ is a d-dimensional ADR *space, and define the integral operator $\Theta = \Theta_E$ for all functions $f \in L^p(E, \sigma)$, $1 \leq p \leq \infty$, by*

$$(\Theta f)(x) := \int_E \theta(x, y) f(y)\, d\sigma(y), \qquad \forall\, x \in \mathscr{X} \setminus E. \tag{1.25}$$

Let $\mathbb{D}(E)$ denote a dyadic cube structure on E and, for each $Q \in \mathbb{D}(E)$, denote by $T_E(Q)$ the dyadic Carleson tent over Q. Finally, let $\rho_\#$ be the regularized version of the quasi-distance ρ as in Theorem 2.2 and, for each $x \in \mathscr{X}$, set $\delta_E(x) := \inf\{\rho_\#(x, y) : y \in E\}$.

Then the following are equivalent:

(1) [L^2 `square function estimate`] *There exists $C \in (0, \infty)$ with the property that for each $f \in L^2(E, \sigma)$ one has*

$$\int_{\mathscr{X} \setminus E} |(\Theta f)(x)|^2 \delta_E(x)^{2\upsilon - (m-d)}\, d\mu(x) \leq C \int_E |f(x)|^2\, d\sigma(x). \tag{1.26}$$

(2) [`Carleson measure condition on dyadic tents for` Θ `tested on` 1] *There holds*

$$\sup_{Q \in \mathbb{D}(E)} \left(\tfrac{1}{\sigma(Q)} \int_{T_E(Q)} \big|\big(\Theta 1\big)(x)\big|^2\, \delta_E(x)^{2\upsilon - (m-d)}\, d\mu(x) \right) < \infty. \tag{1.27}$$

(3) [`Carleson measure condition on dyadic tents for` Θ `acting on` L^∞] *There exists a constant $C \in (0, \infty)$ with the property that for each $f \in L^\infty(E, \sigma)$*

$$\sup_{Q \in \mathbb{D}(E)} \left(\tfrac{1}{\sigma(Q)} \int_{T_E(Q)} |(\Theta f)(x)|^2 \delta_E(x)^{2\upsilon - (m-d)}\, d\mu(x) \right)^{1/2} \leq C \|f\|_{L^\infty(E,\sigma)}. \tag{1.28}$$

(4) [`Carleson measure condition on balls for` Θ `tested on` 1] *There holds*

$$\sup_{x \in E,\, r > 0} \left(\frac{1}{\sigma\big(E \cap B_{\rho_\#}(x, r)\big)} \int_{B_{\rho_\#}(x,r) \setminus E} |\Theta 1|^2 \delta_E^{2\upsilon - (m-d)}\, d\mu \right) < \infty. \tag{1.29}$$

(5) [`Carleson measure condition for` Θ `tested on a para-accretive function`] *There exists a para-accretive function* $b : E \to \mathbb{C}$ *with the property that*

$$\sup_{Q \in \mathbb{D}(E)} \left(\tfrac{1}{\sigma(Q)} \int_{T_E(Q)} |(\Theta b)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \right) < \infty. \tag{1.30}$$

(6) [`Carleson measure condition on balls for` Θ `acting on` L^∞] *There exists* $C \in (0,\infty)$ *with the property that for each* $f \in L^\infty(E,\sigma)$

$$\sup_{x \in E,\, r>0} \left(\frac{1}{\sigma\big(E \cap B_{\rho_\#}(x,r)\big)} \int_{B_{\rho_\#}(x,r)\setminus E} |\Theta f|^2 \delta_E^{2\upsilon-(m-d)}\, d\mu \right)^{1/2} \leq C \|f\|_{L^\infty(E,\sigma)}. \tag{1.31}$$

(7) [`Local` $T(b)$ `condition on dyadic cubes`] *There exist two finite constants* $C_0 \geq 1$ *and* $c_0 \in (0,1]$, *along with a collection* $\{b_Q\}_{Q\in\mathbb{D}(E)}$ *of* σ*-measurable functions* $b_Q : E \to \mathbb{C}$ *such that for each* $Q \in \mathbb{D}(E)$ *the following hold:*

$$\begin{gathered} \int_E |b_Q|^2\, d\sigma \leq C_0 \sigma(Q), \\ \Big| \textstyle\int_{\widetilde{Q}} b_Q\, d\sigma \Big| \geq \tfrac{1}{C_0}\, \sigma(\widetilde{Q}) \quad \textit{for some } \widetilde{Q} \subseteq Q,\ \ \ell(\widetilde{Q}) \geq c_0 \ell(Q), \\ \int_{T_E(Q)} |(\Theta\, b_Q)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \leq C_0 \sigma(Q). \end{gathered} \tag{1.32}$$

(8) [`Local` $T(b)$ `condition on surface balls`] *There exist* $C_0 \in [1,\infty)$ *and, for each surface ball* $\Delta = \Delta(x_o, r) := B_{\rho_\#}(x_o, r) \cap E$, *where* x_o *is a point in* E *and* r *is a finite number in* $\big(0, \mathrm{diam}_\rho(E)\big]$, *a* σ*-measurable function* $b_\Delta : E \to \mathbb{C}$ *supported in* Δ, *such that the following estimates hold:*

$$\begin{gathered} \int_E |b_\Delta|^2\, d\sigma \leq C_0 \sigma(\Delta), \qquad \Big| \int_\Delta b_\Delta\, d\sigma \Big| \geq \frac{1}{C_0}\, \sigma(\Delta), \\ \int_{B_{\rho_\#}(x_o, 2C_\rho r)\setminus E} |(\Theta\, b_\Delta)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \leq C_0 \sigma(\Delta). \end{gathered} \tag{1.33}$$

(9) [`Big Pieces of Square Function Estimate`] *The set* E *has* BPSFE *relative to the kernel* θ *(cf. Definition 4.1).*

(10) [`Iterated Big Pieces of Square Function Estimate`] *The set* E *has* $(\mathrm{BP})^k$SFE *relative to the kernel* θ *(cf. Definition 4.4) for some, or any,* $k \in \mathbb{N}$.

(11) [`Weak-`L^p `square function estimate`] *There exist an exponent* $p \in (0,\infty)$ *and constants* $C, \kappa \in (0,\infty)$ *such that for every* $f \in L^p(E,\sigma)$

$$\sup_{\lambda>0} \left[\lambda \cdot \sigma\Big(\Big\{ x \in E : \int_{\Gamma_\kappa(x)} |(\Theta f)(y)|^2 \frac{d\mu(y)}{\delta_E(y)^{m-2\upsilon}} > \lambda^2 \Big\} \Big)^{1/p} \right] \leq C \Big(\int_E |f|^p\, d\sigma \Big)^{1/p}, \tag{1.34}$$

where $\Gamma_\kappa(x)$ *stands for the nontangential approach region defined in* (6.1).

(12) [`Hardy and` L^p `square function estimates`] *Set* $\gamma := \min\{\alpha, (\log_2 C_\rho)^{-1}\}$. *Then for each* $p \in \big(\frac{d}{d+\gamma}, \infty\big)$ *the operator* Θ *extends to the space* $H^p(E, \rho|_E, \sigma)$, *defined as the Lebesgue space* $L^p(E,\sigma)$ *if* $p \in (1,\infty)$, *and the Coifman-Weiss Hardy space on the space of homogeneous type* $(E, \rho|_E, \sigma)$ *if* $p \in \big(\frac{d}{d+\gamma}, 1\big]$, *and this extension satisfies*

$$\left\| \Big(\int_{\Gamma_\kappa(x)} |(\Theta f)(y)|^2 \, \frac{d\mu(y)}{\delta_E(y)^{m-2\upsilon}} \Big)^{\frac{1}{2}} \right\|_{L^p_x(E,\sigma)} \leq C \|f\|_{H^p(E,\rho|_E,\sigma)}, \tag{1.35}$$

for each function $f \in H^p(E, \rho|_E, \sigma)$.

(13) [`Mixed-norm space estimate`] *For each* $p \in \big(\frac{d}{d+\gamma}, \infty\big)$, *with* $\gamma := \min\{\alpha, (\log_2 C_\rho)^{-1}\}$, *and each* $q \in (1,\infty)$, *the operator*

$$\delta_E^{\upsilon - m/q} \Theta : H^p(E, \rho|_E, \sigma) \longrightarrow L^{(p,q)}(\mathscr{X}, E) \tag{1.36}$$

is well-defined, linear and bounded, where $L^{(p,q)}(\mathscr{X}, E)$ *is the mixed norm space defined in* (6.10).

A few comments pertaining to the nature and scope of Theorem 1.1 are in order.

• Theorem 1.1 makes the case that estimating the square function in L^p, along with other related issues considered above, may be regarded as "zeroth order calculus", since only integrability and quasi-metric geometry are involved, without recourse to differentiability (or vector space structures). In particular, our approach is devoid of any PDE results and techniques. Compared with works in the upper-half space $\mathbb{R}^n \times (0,\infty)$, or so-called generalized upper-half spaces $E \times (0,\infty)$ (cf., e.g., [34] and the references therein), here we work in an ambient $\mathscr{X}$ with no distinguished "vertical" direction. Moreover, the set E is allowed to have arbitrary ADR co-dimension in the ambient $\mathscr{X}$. In this regard we also wish to point out that Theorem 1.1 permits the consideration of fractal subsets of the Euclidean space (such as the case when E is the von Koch's snowflake in $\mathbb{R}^2$, in which scenario $d = \frac{\ln 4}{\ln 3}$).

• Passing from L^2 estimates to L^p estimates is no longer done via a linearization procedure (since the environment no longer permits it) and, instead, we use tent space theory and exploit the connection between the Lusin and the Carleson operators on spaces of homogeneous type (thus generalizing work from [16] in the Euclidean setting). This reinforces the philosophy that the square-function is a singular integral operator at least in spirit (if not in the letter).

• The various quantitative aspects of the claims in items *(1)*-*(11)* of Theorem 1.1 are naturally related to one another. The reader is also alerted to the fact that similar results to those contained in Theorem 1.1 are proved in the body of the manuscript for a larger class of kernels (satisfying less stringent conditions) than in the theorem above. The specific way in which Theorem 1.1 follows from these more general results is discussed in § 7.

We now proceed to describe several consequences of Theorem 1.1 for subsets E of the Euclidean space. First we record the following square function estimate, which extends work from [25].

Theorem 1.2. *Suppose that* E *is a closed subset of* $\mathbb{R}^{n+1}$ *which is* d*-dimensional* ADR *for some* $d \in \{1, \dots, n\}$ *and denote by* σ *the surface measure induced by the* d*-dimensional Hausdorff measure on* E. *Assume that* E *has big pieces of Lipschitz images of subsets of* $\mathbb{R}^d$, *i.e., there*

exist $\varepsilon, M \in (0,\infty)$ so that for every $x \in E$ and every $R \in (0,\infty)$, there is a Lipschitz mapping φ with Lipschitz norm $\leq M$ from the ball $B^d(0,R)$ in $\mathbb{R}^d$ into $\mathbb{R}^{n+1}$ such that

$$\sigma\Big(E \cap B(x,R) \cap \varphi(B^d(0,R))\Big) \leq \varepsilon R^d. \tag{1.37}$$

Suppose $\psi : \mathbb{R}^{n+1} \to \mathbb{R}$ is a compactly supported, smooth, odd function and for each $k \in \mathbb{Z}$ set $\psi_k(x) := 2^{-kd}\psi\big(\frac{x}{2^k}\big)$ for $x \in \mathbb{R}^{n+1}$. Then for every $q \in (1,\infty)$ and every $p \in \big(\frac{d}{d+1},\infty\big)$ there exists $C \in (0,\infty)$ such that

$$\Bigg\| \Big(\sum_{k=-\infty}^{+\infty} ⨍_{y \in \Delta(x,2^k)} \Big| \int_E \psi_k(z-y) f(z)\, d\sigma(z) \Big|^q d\sigma(y) \Big)^{1/q} \Bigg\|_{L^p_x(E,\sigma)} \leq C\|f\|_{H^p(E,\sigma)} \tag{1.38}$$

for every $f \in H^p(E,\sigma)$, where $\Delta(x,2^k) := \{y \in E : |y-x| < 2^k\}$ for each $x \in E$ and $k \in \mathbb{Z}$.

The particular case when $p = q = 2$, in which scenario (1.38) takes the form

$$\sum_{k=-\infty}^{+\infty} \int_E \Big| \int_E \psi_k(x-y) f(y)\, d\sigma(y) \Big|^2 d\sigma(x) \leq C \int_E |f|^2\, d\sigma, \tag{1.39}$$

has been treated in [25, § 3, p. 21]. The main point of Theorem 1.2 is that (1.39) continues to hold, when formulated as in (1.38) for every $p \in \big(\frac{d}{d+1},\infty\big)$. The proof of this result, presented in the last part of § 7, relies on Theorem 1.1 and uses the fact that no regularity condition on the kernel $\theta(x,y)$ is assumed in the variable x (compare with (1.23)-(1.24)).

Next, we discuss another consequence of Theorem 1.1 in the Euclidean setting which gives an extension of results due to G. David and S. Semmes.

Theorem 1.3. *Suppose that K is a real-valued function satisfying*

$$\begin{gathered} K \in C^2(\mathbb{R}^{n+1} \setminus \{0\}), \quad K \text{ is odd, and} \\ K(\lambda x) = \lambda^{-n} K(x) \text{ for all } \lambda > 0,\ x \in \mathbb{R}^{n+1} \setminus \{0\}. \end{gathered} \tag{1.40}$$

Let E be a closed subset of $\mathbb{R}^{n+1}$ which is n-dimensional ADR*, denote by σ the surface measure induced by the n-dimensional Hausdorff measure on E, and define the integral operator $\mathcal{T}$ acting on functions $f \in L^p(E,\sigma)$, $1 \leq p \leq \infty$, by*

$$\mathcal{T}f(x) := \int_E K(x-y) f(y)\, d\sigma(y), \qquad \forall\, x \in \mathbb{R}^{n+1} \setminus E. \tag{1.41}$$

Finally, let $\mathbb{D}(E)$ denote a dyadic cube structure on E and, for each $Q \in \mathbb{D}(E)$, denote by $T_E(Q)$ the dyadic Carleson tent over Q.

*Then, if the set E is actually uniformly rectifiable (*UR*), in the sense of Definition 5.4, conditions (1)-(5) below hold:*

(1) [L^2 `square function estimate`] *There exists $C \in (0,\infty)$ with the property that for each $f \in L^2(E,\sigma)$ one has*

$$\int\limits_{\mathbb{R}^{n+1}\setminus E} |(\nabla \mathcal{T} f)(x)|^2 \operatorname{dist}(x,E)\, dx \leq C \int_E |f(x)|^2\, d\sigma(x). \tag{1.42}$$

(2) [`Carleson measure condition on dyadic tents for` $\mathcal{T}$ `acting on` L^∞] *There exists a constant* $C \in (0,\infty)$ *with the property that for each* $f \in L^\infty(E,\sigma)$

$$\sup_{Q\in\mathbb{D}(E)} \left(\frac{1}{\sigma(Q)} \int_{T_E(Q)} |(\nabla\mathcal{T} f)(x)|^2 \operatorname{dist}(x,E)\,dx \right)^{1/2} \leq C\|f\|_{L^\infty(E,\sigma)}. \tag{1.43}$$

In particular,

$$\sup_{Q\in\mathbb{D}(E)} \left(\frac{1}{\sigma(Q)} \int_{T_E(Q)} |(\nabla\mathcal{T} 1)(x)|^2 \operatorname{dist}(x,E)\,dx \right) < \infty. \tag{1.44}$$

(3) [`Carleson measure condition on balls for` $\mathcal{T}$ `acting on` L^∞] *There exists a constant* $C \in (0,\infty)$ *with the property that for each* $f \in L^\infty(E,\sigma)$

$$\sup_{x\in E,\, r>0} \left(\frac{1}{\sigma\big(E\cap B(x,r)\big)} \int_{B(x,r)\setminus E} |(\nabla\mathcal{T} f)(y)|^2 \operatorname{dist}(y,E)\,dy \right)^{1/2} \leq C\|f\|_{L^\infty(E,\sigma)}. \tag{1.45}$$

In particular,

$$\sup_{x\in E,\, r>0} \left(\frac{1}{\sigma\big(E\cap B(x,r)\big)} \int_{B(x,r)\setminus E} |(\nabla\mathcal{T} 1)(y)|^2 \operatorname{dist}(y,E)\,dy \right) < \infty. \tag{1.46}$$

(4) [`Hardy and` L^p `square function estimates`] *For each* $p \in \big(\frac{n}{n+1},\infty\big)$ *let* $H^p(E,\sigma)$ *stand for the Lebesgue scale* $L^p(E,\sigma)$ *if* $p \in (1,\infty)$*, and the Coifman-Weiss scale of Hardy spaces on the space of homogeneous type* $(E,|\cdot-\cdot|,\sigma)$ *if* $p \in \big(\frac{n}{n+1},1\big]$*. Then the operator* $\mathcal{T}$ *extends to the space* $H^p(E,\sigma)$ *and this extension satisfies*

$$\left\| \Big(\int_{\Gamma_\kappa(x)} |(\nabla\mathcal{T} f)(y)|^2 \frac{dy}{\operatorname{dist}(y,E)^{n-1}} \Big)^{\frac12} \right\|_{L^p_x(E,\sigma)} \leq C\|f\|_{H^p(E,\sigma)}, \quad \forall\, f \in H^p(E,\sigma). \tag{1.47}$$

(5) [`Mixed-norm space estimate`] *For each* $p \in \big(\frac{n}{n+1},\infty\big)$ *and each* $q \in (1,\infty)$ *the operator*

$$\operatorname{dist}(\cdot,E)\,\nabla\mathcal{T} : H^p(E,\sigma) \longrightarrow L^{(p,q)}(\mathbb{R}^{n+1},E) \tag{1.48}$$

is well-defined, linear and bounded, where $L^{(p,q)}(\mathbb{R}^{n+1},E)$ *is the mixed norm space defined in* (6.10) *(corresponding here to* $\mathscr{X} := \mathbb{R}^{n+1}$ *and* $\rho := |\cdot-\cdot|$*).*

Theorem 1.1 particularized to the setting of Theorem 1.3 gives that conditions *(1)*-*(5)* above are equivalent. The fact that *(1)* holds in the special case when $\mathcal{T}$ is associated as in (1.41) with each of the kernels $K_j(x) := x_j/|x|^{n+1}$, $1 \leq j \leq n+1$, is due to David and Semmes [26]. The new result here is that *(1)* (hence also all of *(1)*-*(5)*) holds more generally for the entire class of kernels described in (1.40). We shall prove the latter fact in Corollary 5.7 below. Compared with [25], the class of kernels (1.40) is not tied up to any particular partial differential operator (in the manner that the kernels $K_j(x) := x_j/|x|^{n+1}$, $1 \leq j \leq n+1$, are related to the Laplacian). Moreover, in § 5.3 we establish a version of Theorem 1.3 for variable coefficient kernels, which ultimately applies to integral operators on domains on manifolds

associated with the Schwartz kernels of certain classes of pseudodifferential operators acting between vector bundles.

The condition that the set E is UR in the context of Theorem 1.3 is optimal, as seen from the converse statement stated below. This result is closely interfaced with the characterization of uniform rectifiability, due David and Semmes, in terms of (1.15)-(1.16). In keeping with these conditions, the formulation of our result involves the Riesz-transform operator $\mathcal{R} := \nabla\mathcal{S}$.

Theorem 1.4. *Let E be a closed subset of $\mathbb{R}^{n+1}$ which is n-dimensional* ADR, *denote by σ the surface measure induced by the n-dimensional Hausdorff measure on E, and define the vector-valued integral operator $\mathcal{R}$ acting on functions $f \in L^p(E,\sigma)$, $1 \leq p \leq \infty$, by*

$$\mathcal{R}f(x) := \int_E \frac{x-y}{|x-y|^{n+1}} f(y)\, d\sigma(y), \qquad \forall\, x \in \mathbb{R}^{n+1} \setminus E. \tag{1.49}$$

As before, let $\mathbb{D}(E)$ denote a dyadic cube structure on E and, for each $Q \in \mathbb{D}(E)$, denote by $T_E(Q)$ the dyadic Carleson tent over Q. In this setting, consider the following conditions:

(1) [L^2 `square function estimate`] *There exists $C \in (0,\infty)$ with the property that for each $f \in L^2(E,\sigma)$ one has*

$$\int\limits_{\mathbb{R}^{n+1}\setminus E} |(\nabla\mathcal{R}f)(x)|^2 \operatorname{dist}(x,E)\, dx \leq C \int_E |f(x)|^2\, d\sigma(x). \tag{1.50}$$

(2) [`Carleson measure condition on dyadic tents for` $\mathcal{R}$ `tested on` 1] *There holds*

$$\sup_{Q\in\mathbb{D}(E)} \left(\frac{1}{\sigma(Q)} \int_{T_E(Q)} |(\nabla\mathcal{R}1)(x)|^2 \operatorname{dist}(x,E)\, dx \right) < \infty. \tag{1.51}$$

(3) [`Carleson measure condition on dyadic tents for` $\mathcal{R}$ `acting on` L^∞] *There exists a constant $C \in (0,\infty)$ with the property that for each $f \in L^\infty(E,\sigma)$*

$$\sup_{Q\in\mathbb{D}(E)} \left(\frac{1}{\sigma(Q)} \int_{T_E(Q)} |(\nabla\mathcal{R}f)(x)|^2 \operatorname{dist}(x,E)\, dx \right)^{1/2} \leq C\|f\|_{L^\infty(E,\sigma)}. \tag{1.52}$$

(4) [`Carleson measure condition on balls for` $\mathcal{R}$ `tested on` 1] *There holds*

$$\sup_{x\in E,\, r>0} \left(\frac{1}{\sigma\big(E\cap B(x,r)\big)} \int_{B(x,r)\setminus E} |(\nabla\mathcal{R}1)(y)|^2 \operatorname{dist}(y,E)\, dy \right) < \infty. \tag{1.53}$$

(5) [`Carleson measure condition for` $\mathcal{R}$ `tested on a para-accretive function`] *There exists a para-accretive function $b : E \to \mathbb{C}$ with the property that*

$$\sup_{Q\in\mathbb{D}(E)} \left(\frac{1}{\sigma(Q)} \int_{T_E(Q)} |(\nabla\mathcal{R}b)(x)|^2 \operatorname{dist}(x,E)\, dx \right) < \infty. \tag{1.54}$$

(6) [`Carleson measure condition on balls for` $\mathcal{R}$ `acting on` L^∞] *There exists a constant* $C \in (0,\infty)$ *with the property that for each* $f \in L^\infty(E,\sigma)$

$$\sup_{x\in E,\, r>0}\left(\frac{1}{\sigma\big(E\cap B(x,r)\big)}\int_{B(x,r)\setminus E}|(\nabla\mathcal{R}f)(y)|^2\,\mathrm{dist}\,(y,E)\,dy\right)^{1/2}\leq C\|f\|_{L^\infty(E,\sigma)}. \quad (1.55)$$

(7) [`Local` $T(b)$ `condition on dyadic cubes`] *There exist finite constants* $C_0 \geq 1$, $c_0 \in (0,1]$ *as well as a collection* $\{b_Q\}_{Q\in\mathbb{D}(E)}$ *of* σ*-measurable functions* $b_Q : E \to \mathbb{C}$ *such that for each* $Q \in \mathbb{D}(E)$ *the following hold:*

$$\int_E |b_Q|^2\,d\sigma \leq C_0\sigma(Q),$$

$$\Big|\int_{\widetilde{Q}} b_Q\,d\sigma\Big| \geq \tfrac{1}{C_0}\,\sigma(\widetilde{Q}) \quad \textit{for some } \widetilde{Q}\subseteq Q,\ \ \ell(\widetilde{Q})\geq c_0\ell(Q), \quad (1.56)$$

$$\int_{T_E(Q)}|(\nabla\mathcal{R}b_Q)(x)|^2\,\mathrm{dist}\,(x,E)\,dx \leq C_0\sigma(Q).$$

(8) [`Local` $T(b)$ `condition on surface balls`] *There exist* $C_0 \in [1,\infty)$ *and, for each surface ball* $\Delta = \Delta(x_o,r) := B(x_o,r)\cap E$*, where* x_o *is a point in* E *and* r *is a finite number in* $\big(0,\mathrm{diam}(E)\big]$*, a* σ*-measurable function* $b_\Delta : E \to \mathbb{C}$ *supported in* Δ*, such that the following estimates hold:*

$$\int_E |b_\Delta|^2\,d\sigma \leq C_0\sigma(\Delta), \qquad \Big|\int_\Delta b_\Delta\,d\sigma\Big| \geq \frac{1}{C_0}\,\sigma(\Delta),$$
$$\int_{B(x_o,4r)\setminus E}|(\nabla\mathcal{R}\,b_\Delta)(x)|^2\,\mathrm{dist}\,(x,E)\,dx \leq C_0\sigma(\Delta). \quad (1.57)$$

(9) [`Weak-`L^p `square function estimate`] *There exist an index* $p \in (0,\infty)$ *and constants* $C,\kappa \in (0,\infty)$ *such that for every* $f \in L^p(E,\sigma)$

$$\sup_{\lambda>0}\left[\lambda\cdot\sigma\Big(\Big\{x\in E:\int_{\Gamma_\kappa(x)}\frac{|(\nabla\mathcal{R}f)(y)|^2}{\mathrm{dist}\,(y,E)^{n-1}}\,dy>\lambda^2\Big\}\Big)^{1/p}\right]\leq C\Big(\int_E |f|^p\,d\sigma\Big)^{1/p}, \quad (1.58)$$

where $\Gamma_\kappa(x) := \big\{y\in\mathbb{R}^{n+1}\setminus E:\ |x-y|<(1+\kappa)\,\mathrm{dist}(y,E)\big\}$ *for each* $x \in E$*.*

Then if any of properties (1)-(9) holds it follows that E *is a* UR *set.*

The fact that condition *(1)* above implies that E is a UR set has been proved by David and Semmes (see [26, pp. 252-267]). Based on this result, that *(2)*-*(3)* also imply that E is a UR set then follows with the help of Theorem 1.1 upon observing that the components of $\mathcal{R}$ are operators $\mathcal{T}$ as in (1.41) associated with the kernels $K_j(x) := x_j/|x|^{n+1}$, $j \in \{1,...,n+1\}$, which satisfy (1.23)-(1.24). Compared to David and Semmes' result mentioned above (to the effect that the L^2 square function for the operators associated with the kernels K_j, $1 \leq j \leq n+1$, implies that the set E is UR), a remarkable corollary of Theorem 1.4 is that a mere weak-L^2 square function estimate for the operators associated with the kernels $K_j(x) := x_j/|x|^{n+1}$, $j \in \{1,...,n+1\}$, as in (1.41) implies that E is a UR set.

Throughout the manuscript, we adopt the following conventions. The letter C represents a finite positive constant that may change from one line to the next. The infinity symbol $\infty := +\infty$. The set of positive integers is denoted by $\mathbb{N}$, and the set $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

2 Analysis and Geometry on Quasi-Metric Spaces

This section contains preliminary material, organized into four subsections dealing, respectively, with: a metrization result for arbitrary quasi-metric spaces, geometrically doubling quasi-metric spaces, approximations to the identity, and a discussion of the nature of Carleson tents in quasi-metric spaces.

2.1 A metrization result for general quasi-metric spaces

Here the goal is to review a sharp quantitative metrization result for quasi-metric spaces (cf. Theorem 2.2), and record some useful properties of the Hausdorff outer-measure on quasi-metric spaces (cf. Proposition 2.4). We begin, however, by introducing basic terminology and notation in the definition below.

Definition 2.1. *Assume that $\mathscr{X}$ is a set of cardinality at least two:*

(1) A function $\rho : \mathscr{X} \times \mathscr{X} \to [0,\infty)$ is called a `quasi-distance` *on $\mathscr{X}$ provided there exist two constants $C_0, C_1 \in [1,\infty)$ with the property that for every $x, y, z \in X$, one has*

$$\rho(x,y) = 0 \Leftrightarrow x = y, \quad \rho(y,x) \leq C_0\rho(x,y), \quad \rho(x,y) \leq C_1 \max\{\rho(x,z), \rho(z,y)\}. \quad (2.1)$$

(2) Denote by $\mathfrak{Q}(\mathscr{X})$ the collection of all quasi-distances on $\mathscr{X}$, and call a pair $(\mathscr{X}, \rho)$ a `quasi-metric space` *provided $\rho \in \mathfrak{Q}(\mathscr{X})$. Also, given $\rho \in \mathfrak{Q}(\mathscr{X})$ and $E \subseteq \mathscr{X}$ of cardinality at lest two, denote by $\rho\big|_E \in \mathfrak{Q}(E)$ the restriction of the function ρ to $E \times E$.*

(3) For each $\rho \in \mathfrak{Q}(\mathscr{X})$, define C_ρ to be the smallest constant which can play the role of C_1 in the last inequality in (2.1)*, i.e.,*

$$C_\rho := \sup_{\substack{x,y,z\in\mathscr{X} \\ \text{not all equal}}} \frac{\rho(x,y)}{\max\{\rho(x,z), \rho(z,y)\}} \in [1,\infty), \quad (2.2)$$

and define $\widetilde{C}_\rho$ to be the smallest constant which can play the role of C_0 in the first inequality in (2.1)*, i.e.,*

$$\widetilde{C}_\rho := \sup_{\substack{x,y\in\mathscr{X} \\ x\neq y}} \frac{\rho(y,x)}{\rho(x,y)} \in [1,\infty). \quad (2.3)$$

(4) Given $\rho \in \mathfrak{Q}(\mathscr{X})$, define the ρ-`ball` *(or, simply* `ball` *if the quasi-distance ρ is clear from the context) centered at $x \in \mathscr{X}$ with radius $r \in (0,\infty)$ to be*

$$B_\rho(x,r) := \{y \in \mathscr{X} : \rho(x,y) < r\}. \quad (2.4)$$

Also, call $E \subseteq \mathscr{X}$ ρ-`bounded` *if E is contained in a ρ-ball, and define its ρ-*`diameter` *(or, simply,* `diameter`*) as*

$$\mathrm{diam}_\rho(E) := \sup\big\{\rho(x,y) : x,y \in E\big\}. \quad (2.5)$$

The ρ-`distance` *(or, simply* `distance`*) between two arbitrary, nonempty sets $E, F \subseteq \mathscr{X}$ is naturally defined as*

$$\mathrm{dist}_\rho(E,F) := \inf\{\rho(x,y) : x \in E,\ y \in F\}. \quad (2.6)$$

If $E = \{x\}$ for some $x \in \mathscr{X}$ and $F \subseteq \mathscr{X}$, abbreviate $\mathrm{dist}_\rho(x,F) := \mathrm{dist}_\rho(\{x\},F)$.

(5) *Given a quasi-distance $\rho \in \mathfrak{Q}(\mathscr{X})$ define τ_ρ, the* `topology canonically induced by` ρ `on` $\mathscr{X}$, *to be the largest topology on $\mathscr{X}$ with the property that for each point $x \in \mathscr{X}$ the family $\{B_\rho(x,r)\}_{r>0}$ is a fundamental system of neighborhoods of x.*

(6) *Call two functions $\rho_1, \rho_2 : \mathscr{X} \times \mathscr{X} \to [0,\infty)$* `equivalent`, *and write $\rho_1 \approx \rho_2$, if there exist $C', C'' \in (0,\infty)$ with the property that*

$$C'\rho_1 \le \rho_2 \le C''\rho_1 \quad \text{on } \mathscr{X} \times \mathscr{X}. \tag{2.7}$$

A few comments are in order. Suppose that $(\mathscr{X}, \rho)$ is a quasi-metric space. It is then clear that if $\rho' : \mathscr{X} \times \mathscr{X} \to [0,\infty)$ is such that $\rho' \approx \rho$ then $\rho' \in \mathfrak{Q}(\mathscr{X})$ and $\tau_{\rho'} = \tau_\rho$. Also, it may be checked that

$$\mathcal{O} \in \tau_\rho \Longleftrightarrow \mathcal{O} \subseteq \mathscr{X} \text{ and } \forall\, x \in \mathcal{O}\ \exists\, r > 0 \text{ such that } B_\rho(x,r) \subseteq \mathcal{O}. \tag{2.8}$$

As is well-known, the topology induced by the given quasi-distance on a quasi-metric space is metrizable. Below we shall review a recent result proved in [60] which is an optimal quantitative version of this fact, and which sharpens earlier work from [56]. To facilitate the subsequent discussion we first make a definition. Assume that $\mathscr{X}$ is an arbitrary, nonempty set. Given an arbitrary function $\rho : \mathscr{X} \times \mathscr{X} \to [0,\infty)$ define its `symmetrization` ρ_{sym} as

$$\rho_{sym} : \mathscr{X} \times \mathscr{X} \longrightarrow [0,\infty), \qquad \rho_{sym}(x,y) := \max\{\rho(x,y), \rho(y,x)\}, \quad \forall\, x,y \in \mathscr{X}. \tag{2.9}$$

Then ρ_{sym} is symmetric, i.e., $\rho_{sym}(x,y) = \rho_{sym}(y,x)$ for every $x,y \in \mathscr{X}$, and $\rho_{sym} \ge \rho$ on $\mathscr{X} \times \mathscr{X}$. In fact, ρ_{sym} is the smallest nonnegative function defined on $\mathscr{X} \times \mathscr{X}$ which is symmetric and pointwise $\ge \rho$. Furthermore, if $(\mathscr{X}, \rho)$ is a quasi-metric space then

$$\rho_{sym} \in \mathfrak{Q}(\mathscr{X}), \quad C_{\rho_{sym}} \le C_\rho, \quad \widetilde{C}_{\rho_{sym}} = 1, \quad \text{and} \quad \rho \le \rho_{sym} \le \widetilde{C}_\rho\, \rho. \tag{2.10}$$

Here is the quantitative metrization theorem from [60] alluded to above.

Theorem 2.2. *Let $(\mathscr{X}, \rho)$ be a quasi-metric space and assume that $C_\rho, \widetilde{C}_\rho \in [1,\infty)$ are as in (2.2)-(2.3). Introduce*

$$\alpha_\rho := \frac{1}{\log_2 C_\rho} \in (0,\infty], \tag{2.11}$$

and define the regularization $\rho_\# : \mathscr{X} \times \mathscr{X} \to [0,\infty)$ of ρ as follows. When $\alpha_\rho < \infty$, for each $x, y \in \mathscr{X}$ set

$$\begin{aligned} \rho_\#(x,y) := \inf\Big\{ \Big(\sum_{i=1}^{N} \rho_{sym}(\xi_i, \xi_{i+1})^{\alpha_\rho}\Big)^{\frac{1}{\alpha_\rho}} : N \in \mathbb{N} \text{ and } \xi_1, \dots, \xi_{N+1} \in \mathscr{X}, \\ \text{(not necessarily distinct) such that } \xi_1 = x \text{ and } \xi_{N+1} = y \Big\}, \end{aligned} \tag{2.12}$$

while if $\alpha_\rho = \infty$ then for each $x, y \in \mathscr{X}$ set

$$\begin{aligned} \rho_\#(x,y) := \inf\Big\{ \max_{1 \le i \le N} \rho_{sym}(\xi_i, \xi_{i+1}) : N \in \mathbb{N} \text{ and } \xi_1, \dots, \xi_{N+1} \in \mathscr{X}, \\ \text{(not necessarily distinct) such that } \xi_1 = x \text{ and } \xi_{N+1} = y \Big\}. \end{aligned} \tag{2.13}$$

Then the following properties hold:

(1) The function $\rho_\#$ is a symmetric quasi-distance on $\mathscr{X}$ and $\rho_\# \approx \rho$. More specifically,

$$(C_\rho)^{-2}\rho(x,y) \leq \rho_\#(x,y) \leq \widetilde{C}_\rho\,\rho(x,y), \quad \forall\, x,y \in \mathscr{X}. \tag{2.14}$$

In particular, $\tau_{\rho_\#} = \tau_\rho$. Also, $C_{\rho_\#} \leq C_\rho$. Furthermore, for any nonempty set E of $\mathscr{X}$, there holds

$$(\rho|_E)_\# \approx \rho|_E \approx (\rho_\#)\big|_E. \tag{2.15}$$

(2) For each finite number $\beta \in (0, \alpha_\rho]$, the function

$$d_{\rho,\beta} : \mathscr{X} \times \mathscr{X} \to [0,\infty), \qquad d_{\rho,\beta}(x,y) := \big[\rho_\#(x,y)\big]^\beta, \qquad \forall\, x,y \in \mathscr{X}, \tag{2.16}$$

is a distance on $\mathscr{X}$, and has the property that $(d_{\rho,\beta})^{1/\beta} \approx \rho$. In particular, $d_{\rho,\beta}$ induces the same topology on $\mathscr{X}$ as ρ, hence τ_ρ is metrizable.

(3) For each finite number $\beta \in (0, \alpha_\rho]$, the function $\rho_\#$ satisfies the following Hölder-type regularity condition of order β (in both variables, simultaneously) on $\mathscr{X} \times \mathscr{X}$:

$$\begin{aligned} &\big|\rho_\#(x,y) - \rho_\#(z,w)\big| \\ &\qquad \leq \tfrac{1}{\beta} \max\big\{\rho_\#(x,y)^{1-\beta}, \rho_\#(z,w)^{1-\beta}\big\}\big(\big[\rho_\#(x,z)\big]^\beta + \big[\rho_\#(y,w)\big]^\beta\big) \end{aligned} \tag{2.17}$$

whenever $x,y,z,w \in \mathscr{X}$ (with the understanding that when $\beta \geq 1$ one also imposes the conditions $x \neq y$ and $z \neq w$). In particular,

$$\rho_\# : \mathscr{X} \times \mathscr{X} \longrightarrow [0,\infty) \quad \textit{is continuous}, \tag{2.18}$$

when $\mathscr{X} \times \mathscr{X}$ is equipped with the natural product topology $\tau_\rho \times \tau_\rho$.

(4) If E is a nonempty subset of $(\mathscr{X}, \tau_\rho)$, then the regularized distance function

$$\delta_E := \mathrm{dist}_{\rho_\#}(\cdot, E) : \mathscr{X} \longrightarrow [0,\infty) \tag{2.19}$$

is equivalent to $\mathrm{dist}_\rho(\cdot, E)$. Furthermore, δ_E is locally Hölder of order β on $\mathscr{X}$ for every $\beta \in (0, \min\{1, \alpha_\rho\}]$, in the sense that there exists $C \in (0,\infty)$ which depends only on $C_\rho, \widetilde{C}_\rho$ and β such that

$$\frac{|\delta_E(x) - \delta_E(y)|}{\rho(x,y)^\beta} \leq C\Big(\rho(x,y) + \max\{\mathrm{dist}_\rho(x,E)\,,\,\mathrm{dist}_\rho(y,E)\}\Big)^{1-\beta} \tag{2.20}$$

for all $x,y \in \mathscr{X}$ with $x \neq y$. In particular,

$$\delta_E : (\mathscr{X}, \tau_\rho) \longrightarrow [0,\infty) \quad \textit{is continuous.} \tag{2.21}$$

The key feature of the result discussed in Theorem 2.2 is the fact that if $(\mathscr{X}, \rho)$ is any quasi-metric space then ρ^β is equivalent to a genuine distance on $\mathscr{X}$ for any finite number $\beta \in (0, (\log_2 C_\rho)^{-1}]$. This result is sharp and improves upon an earlier version due to R.A. Macías and C. Segovia [56], in which these authors have identified a smaller, non-optimal upper-bound for the exponent β.

In anticipation of briefly reviewing the notion of Hausdorff outer-measure on a quasi-metric space, we recall a couple of definitions from measure theory. Specifically, given an outer-measure μ^* on an arbitrary set $\mathscr{X}$, consider the collection of all μ^*-measurable sets defined as

$$\mathfrak{M}_{\mu^*} := \{A \subseteq \mathscr{X} : \mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y \setminus A),\ \forall\, Y \subseteq \mathscr{X}\}. \tag{2.22}$$

Carathéodory's classical theorem allows one to pass from a given outer-measure μ^* on $\mathscr{X}$ to a genuine measure by observing that

$$\mathfrak{M}_{\mu^*} \text{ is a sigma-algebra, and } \mu^*\big|_{\mathfrak{M}_{\mu^*}} \text{ is a complete measure.} \tag{2.23}$$

The restriction of an outer-measure μ^* on $\mathscr{X}$ to a subset E of $\mathscr{X}$, denoted by $\mu^*\lfloor E$, is defined naturally by restricting the function μ^* to the collection of all subsets of E. We shall use the same symbol, $\lfloor$, in denoting the restriction of a measure to a measurable set. In this regard, it is useful to know when the measure associated with the restriction of an outer-measure to a set coincides with the restriction to that set of the measure associated with the given outer-measure. Specifically, it may be checked that if μ^* is an outer-measure on $\mathscr{X}$, then

$$\big(\mu^*\lfloor E\big)\Big|_{\mathfrak{M}_{(\mu^*\lfloor E)}} = \big(\mu^*\big|_{\mathfrak{M}_{\mu^*}}\big)\lfloor E, \qquad \forall\, E \in \mathfrak{M}_{\mu^*}. \tag{2.24}$$

Next, if $(\mathscr{X}, \tau)$ is a topological space and μ^* is an outer-measure on $\mathscr{X}$ such that $\mathfrak{M}_{\mu^*}$ contains the Borel sets in $(\mathscr{X}, \tau)$, then call μ^* a `Borel outer-measure` on $\mathscr{X}$. Furthermore, call such a Borel outer-measure μ^* a `Borel regular outer-measure` if

$$\forall\, A \subseteq \mathscr{X}\ \ \exists \text{ a Borel set } B \text{ in } (\mathscr{X}, \tau) \text{ such that } A \subseteq B \text{ and } \mu^*(A) = \mu^*(B). \tag{2.25}$$

After this digression, we now proceed to introduce the concept of d-dimensional Hausdorff outer-measure for a subset of a quasi-metric space.

Definition 2.3. *Let $(\mathscr{X}, \rho)$ be a quasi-metric space and fix $d \geq 0$. Given a set $A \subseteq \mathscr{X}$, for every $\varepsilon > 0$ define $\mathcal{H}^d_{\mathscr{X},\rho,\varepsilon}(A) \in [0, \infty]$ by setting*

$$\mathcal{H}^d_{\mathscr{X},\rho,\varepsilon}(A) := \inf\Big\{\sum_{j=1}^{\infty} (\mathrm{diam}_\rho(A_j))^d : A \subseteq \bigcup_{j=1}^{\infty} A_j \text{ and } \mathrm{diam}_\rho(A_j) \leq \varepsilon \text{ for every } j\Big\} \tag{2.26}$$

(with the convention that $\inf \emptyset := +\infty$*), then take*

$$\mathcal{H}^d_{\mathscr{X},\rho}(A) := \lim_{\varepsilon \to 0^+} \mathcal{H}^d_{\mathscr{X},\rho,\varepsilon}(A) = \sup_{\varepsilon > 0} \mathcal{H}^d_{\mathscr{X},\rho,\varepsilon}(A) \in [0, \infty]. \tag{2.27}$$

The quantity $\mathcal{H}^d_{\mathscr{X},\rho}(A)$ is called the d-`dimensional Hausdorff outer-measure` *in $(\mathscr{X}, \rho)$ of the set A. Whenever the choice of the quasi-distance ρ is irrelevant or clear from the context, $\mathcal{H}^d_{\mathscr{X},\rho}(A)$ is abbreviated as $\mathcal{H}^d_{\mathscr{X}}(A)$.*

It is readily verified that $\mathcal{H}^0_{\mathscr{X},\rho}$ is equivalent to the counting measure. Other basic properties of the Hausdorff outer-measure are collected in the proposition below, proved in [61]. To state it, recall that a measure μ on a quasi-metric space $(\mathscr{X}, \rho)$ is called `Borel regular` provided it is Borel on $(\mathscr{X}, \tau_\rho)$ and

$$\forall\, \mu\text{-measurable } A \subseteq \mathscr{X}\ \ \exists \text{ a Borel set } B \text{ in } (\mathscr{X}, \tau_\rho) \text{ such that } A \subseteq B \text{ and } \mu(A) = \mu(B). \tag{2.28}$$

Also, we make the convention that, given a quasi-metric space $(\mathscr{X}, \rho)$ and $d \geq 0$,

$$\mathscr{H}^d_{\mathscr{X},\rho} \text{ denotes the } \textit{measure} \text{ associated with the outer-measure } \mathcal{H}^d_{\mathscr{X},\rho} \text{ as in (2.23).} \tag{2.29}$$

Proposition 2.4. *Let $(\mathscr{X},\rho)$ be a quasi-metric space and fix $d \geq 0$. Then the following properties hold:*

(1) $\mathcal{H}^d_{\mathscr{X},\rho}$ is a Borel outer-measure on $(\mathscr{X},\tau_\rho)$. In particular, $\mathscr{H}^d_{\mathscr{X},\rho}$ (introduced in (2.29)) is a Borel measure on $(\mathscr{X},\tau_\rho)$.

(2) If $\rho_\#$ is as in Theorem 2.2 then $\mathcal{H}^d_{\mathscr{X},\rho_\#}$ is actually a Borel regular outer-measure on $(\mathscr{X},\tau_\rho)$. Moreover, $\mathscr{H}^d_{\mathscr{X},\rho_\#}$ is a Borel regular measure on $(\mathscr{X},\tau_\rho)$.

(3) One has $\mathcal{H}^d_{\mathscr{X},\rho'} \approx \mathcal{H}^d_{\mathscr{X},\rho}$ whenever $\rho' \approx \rho$, in the sense that there exist two finite constants $C_1, C_2 > 0$, which depend only on ρ and ρ', such that

$$C_1\,\mathcal{H}^d_{\mathscr{X},\rho}(A) \leq \mathcal{H}^d_{\mathscr{X},\rho'}(A) \leq C_2\,\mathcal{H}^d_{\mathscr{X},\rho}(A) \quad \text{for all } A \subseteq \mathscr{X}. \tag{2.30}$$

(4) Let $E \subseteq \mathscr{X}$ and consider the quasi-metric space $(E,\rho|_E)$. Then the d-dimensional Hausdorff outer-measure in $(E,\rho|_E)$ is equivalent to the restriction to E of the d-dimensional Hausdorff outer-measure in $\mathscr{X}$. That is, in the sense of (2.30),

$$\mathcal{H}^d_{E,\rho|_E} \approx \mathcal{H}^d_{\mathscr{X},\rho}\lfloor E. \tag{2.31}$$

(5) For any $E \subseteq \mathscr{X}$, $\mathcal{H}^d_{\mathscr{X},\rho_\#}\lfloor E$ is a Borel regular outer-measure on $(E,\tau_{\rho|_E})$, and the measure associated with it (as in (2.23)) is a Borel regular measure on $(E,\tau_{\rho|_E})$.

Furthermore, if E is $\mathcal{H}^d_{\mathscr{X},\rho_\#}$-measurable (in the sense of (2.22); hence, in particular, if E is a Borel subset of $(\mathscr{X},\tau_\rho)$), then $\mathscr{H}^d_{\mathscr{X},\rho_\#}\lfloor E$ is a Borel regular measure on $(E,\tau_{\rho|_E})$ and it coincides with the measure associated with the outer-measure $\mathcal{H}^d_{\mathscr{X},\rho_\#}\lfloor E$.

(6) Assume that $m \in (d,\infty)$. Then for each $E \subseteq \mathscr{X}$ one has

$$\mathcal{H}^d_{\mathscr{X},\rho}(E) < \infty \Longrightarrow \mathcal{H}^m_{\mathscr{X},\rho}(E) = 0. \tag{2.32}$$

2.2 Geometrically doubling quasi-metric spaces

In this subsection we shall work in a more specialized setting than that of general quasi-metric spaces considered so far, by considering geometrically doubling quasi-metric spaces, as described in the definition below.

Definition 2.5. *A quasi-metric space $(\mathscr{X},\rho)$ is called* `geometrically doubling` *if there exists a number $N \in \mathbb{N}$, called the geometric doubling constant of $(\mathscr{X},\rho)$, with the property that any ρ-ball of radius r in $\mathscr{X}$ may be covered by at most N ρ-balls in $\mathscr{X}$ of radii $r/2$.*

To put this matter into a larger perspective, recall that a subset E of a quasi-metric space $(\mathscr{X},\rho)$ is said to be `totally bounded` provided that for any $r \in (0,\infty)$ there exists a finite covering of E with ρ-balls of radii r. Then for a quasi-metric space $(\mathscr{X},\rho)$ the quality of being geometrically doubling may be regarded as a scale-invariant version of the demand that all ρ-balls in $\mathscr{X}$ are totally bounded. In fact it may be readily verified that if $(\mathscr{X},\rho)$ is a geometrically doubling quasi-metric space, then

$$\begin{array}{l}\exists\, N \in \mathbb{N} \text{ such that } \forall\, \vartheta \in (0,1) \text{ any } \rho\text{-ball of radius } r \text{ in } \mathscr{X} \\ \text{may be covered by at most } N^{-[\log_2 \vartheta]}\ \rho\text{-balls in } \mathscr{X} \text{ of radii } \vartheta r,\end{array} \tag{2.33}$$

where $[\log_2 \vartheta]$ is the smallest integer greater than or equal to $\log_2 \vartheta$. En route, let us also point out that *the property of being geometrically doubling is hereditary*, in the sense that if $(\mathscr{X}, \rho)$ is a geometrically doubling quasi-metric space with geometric doubling constant N, and if E is an arbitrary subset of $\mathscr{X}$, then $(E, \rho\big|_E)$ is a geometrically doubling quasi-metric space with geometric doubling constant at most equal to $N^{\log_2 C_\rho} N$.

The relevance of the property (of a quasi-metric space) of being geometrically doubling is apparent from the fact that in such a context a number of useful geometrical results hold, which are akin to those available in the Euclidean setting. A case in point, is the Whitney decomposition theorem discussed in Proposition 2.6 below. A version of the classical Whitney decomposition theorem in the Euclidean setting (as presented in, e.g., [69, Theorem 1.1, p. 167]) has been worked out in [18, Theorem 3.1, p. 71] and [19, Theorem 3.2, p. 623] in the context of bounded open sets in spaces of homogeneous type. Recently, the scope of this work has been further refined in [60] by allowing arbitrary open sets in geometrically doubling quasi-metric spaces, as presented in the following proposition.

Proposition 2.6. *Let $(\mathscr{X}, \rho)$ be a geometrically doubling quasi-metric space. Then for each number $\lambda \in (1, \infty)$ there exist constants $\Lambda \in (\lambda, \infty)$ and $N \in \mathbb{N}$, both depending only on $C_\rho, \widetilde{C}_\rho, \lambda$ and the geometric doubling constant of $(\mathscr{X}, \rho)$, and which have the following significance.*

For each open, nonempty, proper subset $\mathcal{O}$ of the topological space $(\mathscr{X}, \tau_\rho)$ there exist an at most countable family of points $\{x_j\}_{j \in J}$ in $\mathcal{O}$ along with a family of real numbers $r_j > 0$, $j \in J$, for which the following properties are valid:

(1) $\mathcal{O} = \bigcup_{j \in J} B_\rho(x_j, r_j)$;

(2) $\sum_{j \in J} \mathbf{1}_{B_\rho(x_j, \lambda r_j)} \leq N$ *on $\mathcal{O}$. In fact, there exists $\varepsilon \in (0, 1)$, which depends only on C_ρ, λ and the geometric doubling constant of $(\mathscr{X}, \rho)$, with the property that for any $x_o \in \mathcal{O}$*

$$\# \Big\{ j \in J : \, B_\rho\big(x_o, \varepsilon \operatorname{dist}_\rho(x_o, \mathscr{X} \setminus \mathcal{O})\big) \cap B_\rho(x_j, \lambda r_j) \neq \emptyset \Big\} \leq N. \tag{2.34}$$

(3) $B_\rho(x_j, \lambda r_j) \subseteq \mathcal{O}$ *and* $B_\rho(x_j, \Lambda r_j) \cap \big[\mathscr{X} \setminus \mathcal{O}\big] \neq \emptyset$ *for every* $j \in J$.

(4) $r_i \approx r_j$ *uniformly for $i, j \in J$ such that $B_\rho(x_i, \lambda r_i) \cap B_\rho(x_j, \lambda r_j) \neq \emptyset$, and there exists a finite constant $C > 0$ with the property that $r_j \leq C \operatorname{diam}_\rho(\mathcal{O})$ for each $j \in J$.*

Regarding terminology, we shall frequently employ the following convention:

Convention 2.7. *Given a geometrically doubling quasi-metric space $(\mathscr{X}, \rho)$, an open, nonempty, proper subset $\mathcal{O}$ of $(\mathscr{X}, \tau_\rho)$, and a parameter $\lambda \in (1, \infty)$, we will refer to the balls $B_{\rho_\#}(x_j, r_j)$ obtained by treating $(\mathscr{X}, \rho_\#)$ in Proposition 2.6 as* `Whitney cubes`*, denote the collection of these cubes by $\mathbb{W}_\lambda(\mathcal{O})$, and for each $I \in \mathbb{W}_\lambda(\mathcal{O})$, write $\ell(I)$ for the* `radius of` *I.*

Furthermore, if $I \in \mathbb{W}_\lambda(\mathcal{O})$ and $c \in (0, \infty)$, we shall denote by cI the `dilate of the cube` *I* `by factor` *c, i.e., the ball having the same center as I and radius $c\ell(I)$.*

Spaces of homogeneous type, reviewed next, are an important subclass of the class of geometrically doubling quasi-metric spaces.

Definition 2.8. *A* `space of homogeneous type` *is a triplet* $(\mathscr{X}, \rho, \mu)$*, where* $(\mathscr{X}, \rho)$ *is a quasi-metric space and* μ *is a Borel measure on* $(\mathscr{X}, \tau_\rho)$ *with the property that all* ρ*-balls are* μ*-measurable, and which satisfies the doubling condition*

$$0 < \mu\big(B_\rho(x, 2r)\big) \leq C\mu\big(B_\rho(x, r)\big) < \infty, \quad \forall\, x \in \mathscr{X}, \ \forall\, r > 0, \tag{2.35}$$

for some finite constant $C \geq 1$*.*

In the context of the above definition, call the number

$$C_\mu := \sup_{x\in\mathscr{X}, r>0} \frac{\mu\big(B_\rho(x, 2r)\big)}{\mu\big(B_\rho(x, r)\big)} \in [1, \infty) \tag{2.36}$$

the `doubling constant` of μ. Iterating (2.35) then gives

$$\begin{gathered} \tfrac{\mu(B_1)}{\mu(B_2)} \leq C_{\mu,\rho}\Big(\frac{\text{radius of } B_1}{\text{radius of } B_2}\Big)^{D_\mu}, \qquad \text{for all } \rho\text{-balls } \ B_2 \subseteq B_1, \\ \text{where } \ D_\mu := \log_2 C_\mu \geq 0 \ \text{ and } \ C_{\mu,\rho} := C_\mu\big(C_\rho \widetilde{C}_\rho\big)^{D_\mu} \geq 1. \end{gathered} \tag{2.37}$$

The exponent D_μ is referred to as the `doubling order` of μ. For further reference, let us also record here the well-known fact that

$$\begin{gathered} \text{given a space of homogeneous type } (\mathscr{X}, \rho, \mu), \text{ one has} \\ \operatorname{diam}_\rho(\mathscr{X}) < \infty \text{ if and only if } \mu(\mathscr{X}) < \infty. \end{gathered} \tag{2.38}$$

Going further, a distinguished subclass of the class of spaces of homogeneous type, which is going to play a basic role in this work, is the category of Ahlfors-David regular spaces defined next.

Definition 2.9. *Suppose that* $d > 0$*. A* d`-dimensional Ahlfors-David regular` *(or, simply,* d*-dimensional* ADR*, or* d-ADR*)* `space` *is a triplet* $(\mathscr{X}, \rho, \mu)$*, where* $(\mathscr{X}, \rho)$ *is a quasi-metric space and* μ *is a Borel measure on* $(\mathscr{X}, \tau_\rho)$ *with the property that all* ρ*-balls are* μ*-measurable, and for which there exists a constant* $C \in [1, \infty)$ *such that*

$$C^{-1}\, r^d \leq \mu\big(B_\rho(x, r)\big) \leq C\, r^d, \quad \forall\, x \in \mathscr{X}, \ \textit{ for every finite } \ r \in (0, \operatorname{diam}_\rho(\mathscr{X})]. \tag{2.39}$$

The constant C *in* (2.39) *will be referred to as the* ADR *constant of* $\mathscr{X}$*.*

As alluded to earlier, if $(\mathscr{X}, \rho, \mu)$ is a d-dimensional ADR space then, trivially, $(\mathscr{X}, \rho, \mu)$ is also a space of homogeneous type. For further reference let us also note here that (cf., e.g., [61])

$$(\mathscr{X}, \rho, \mu) \text{ is } d\text{-ADR} \implies \big(\mathscr{X}, \rho_\#, \mathscr{H}^d_{\mathscr{X}, \rho_\#}\big) \text{ is } d\text{-ADR}. \tag{2.40}$$

In particular, it follows from (2.40), (2.15), and parts *(3)-(5)* in Proposition 2.4 that

$$\left.\begin{array}{r} (\mathscr{X}, \rho) \text{ quasi-metric space,} \\ E \text{ Borel subset of } (\mathscr{X}, \tau_\rho) \\ \sigma \text{ Borel measure on } (E, \tau_{\rho|_E}) \\ \text{such that } (E, \rho|_E, \sigma) \text{ is } d\text{-ADR} \end{array}\right\} \implies \big(E, \rho_\#\big|_E, \mathscr{H}^d_{\mathscr{X}, \rho_\#} \lfloor E\big) \text{ is } d\text{-ADR}. \tag{2.41}$$

Also, if $(\mathscr{X},\rho,\mu)$ is d-ADR, then there exists a finite constant $C>0$ such that

$$\mathcal{H}^d_{\mathscr{X},\rho_\#}(A)\leq C\inf_{\mathcal{O}\text{ open},\,A\subseteq\mathcal{O}}\mu(\mathcal{O})\quad\text{for every } A\subseteq\mathscr{X},\ \text{ and}\tag{2.42}$$

$$\mu(A)\leq C\mathscr{H}^d_{\mathscr{X},\rho_\#}(A)\quad\text{for every Borel subset } A \text{ of } (\mathscr{X},\tau_\rho).\tag{2.43}$$

In addition, if μ is actually a Borel regular measure, then

$$\mu(A)\approx\mathscr{H}^d_{\mathscr{X},\rho_\#}(A),\qquad\text{uniformly for Borel subsets } A \text{ of } (\mathscr{X},\tau_\rho).\tag{2.44}$$

We now discuss a couple of technical lemmas which are going to be useful for us later on.

Lemma 2.10. *Let $(\mathscr{X},\rho,\mu)$ be an m-dimensional* ADR *space for some $m\in(0,\infty)$ and suppose that E is a Borel subset of $(\mathscr{X},\tau_\rho)$ with the property that there exists a Borel measure σ on $(E,\tau_{\rho|_E})$ such that $(E,\rho\big|_E,\sigma)$ is a d-dimensional* ADR *space for some $d\in(0,m)$. Then $\mu(E)=0$.*

Proof. Fix $x\in E$. Using (2.43), (2.41) and item *(6)* in Proposition 2.4, we obtain

$$\mu(E)\leq C\mathscr{H}^m_{\mathscr{X},\rho_\#}(E)=C\lim_{n\to\infty}\mathscr{H}^m_{\mathscr{X},\rho_\#}(E\cap B_{\rho_\#}(x,n))=0,\tag{2.45}$$

since $\mathscr{H}^d_{\mathscr{X},\rho_\#}(E\cap B_{\rho_\#}(x,n))\leq Cn^d<\infty$ for all $n\in\mathbb{N}$. $\square$

Lemma 2.11. *Let $(\mathscr{X},\rho)$ be a quasi-metric space. Suppose that E is a Borel subset of $(\mathscr{X},\tau_\rho)$ such that there exists a Borel measure σ on $(E,\tau_{\rho|_E})$ with the property that $(E,\rho\big|_E,\sigma)$ is a d-dimensional* ADR *space for some $d\in(0,\infty)$. Then there exists a constant $c\in(0,\infty)$ such that*

$$\begin{gathered}\forall\,x\in\mathscr{X},\ \forall\,r\in(0,\operatorname{diam}_{\rho_\#}(E)]\ \textit{with } B_{\rho_\#}(x,r)\cap E\neq\emptyset\ \textit{there holds}\\ \mathscr{H}^d_{\mathscr{X},\rho_\#}\big(B_{\rho_\#}(x,C_\rho r)\cap E\big)\geq c\,r^d.\end{gathered}\tag{2.46}$$

Proof. Fix a point $x\in\mathscr{X}$ with the property that $B_{\rho_\#}(x,r)\cap E\neq\emptyset$. If we now select $y\in B_{\rho_\#}(x,r)\cap E$ then $B_{\rho_\#}(y,r)\subseteq B_{\rho_\#}(x,C_\rho r)$. Recall (2.41) and let C be the ADR constant of $\big(E,\rho_\#\big|_E,\mathscr{H}^d_{\mathscr{X},\rho_\#}\lfloor E\big)$. Then, since $y\in E$,

$$\mathscr{H}^d_{\mathscr{X},\rho_\#}\big(B_{\rho_\#}(x,C_\rho r)\cap E\big)\geq\mathscr{H}^d_{\mathscr{X},\rho_\#}\big(B_{\rho_\#}(y,r)\cap E\big)\geq C^{-1}r^d.\tag{2.47}$$

Hence (2.46) holds with $c:=C^{-1}$. $\square$

Following work in [14] and [22], we now discuss the existence of a `dyadic grid structure` on geometrically doubling quasi-metric spaces. The following result is essentially due to M. Christ [14], with two refinements. First, Christ's dyadic grid result is established in the presence of a background doubling, Borel regular measure, which is more restrictive than merely assuming that the ambient quasi-metric space is geometrically doubling. Second, Christ's dyadic grid result involves a scale $\delta\in(0,1)$ which we here show may be taken to be $\frac{1}{2}$, as in the Euclidean setting.

Proposition 2.12. *Assume that (E,ρ) is a geometrically doubling quasi-metric space and select $\kappa_E \in \mathbb{Z}\cup\{-\infty\}$ with the property that*

$$2^{-\kappa_E-1} < \operatorname{diam}_\rho(E) \leq 2^{-\kappa_E}. \tag{2.48}$$

Then there exist finite constants $a_1 \geq a_0 > 0$ such that for each $k\in\mathbb{Z}$ with $k\geq\kappa_E$, there exists a collection $\mathbb{D}_k(E) := \{Q_\alpha^k\}_{\alpha\in I_k}$ of subsets of E indexed by a nonempty, at most countable set of indices I_k, as well as a family $\{x_\alpha^k\}_{\alpha\in I_k}$ of points in E, such that the collection of all `dyadic cubes` *in E, i.e.,*

$$\mathbb{D}(E) := \bigcup_{k\in\mathbb{Z},\, k\geq\kappa_E} \mathbb{D}_k(E), \tag{2.49}$$

has the following properties:

(1) [All dyadic cubes are open]
For each $k\in\mathbb{Z}$ with $k\geq\kappa_E$ and each $\alpha\in I_k$, the set Q_α^k is open in τ_ρ;

(2) [Dyadic cubes are mutually disjoint within the same generation]
For each $k\in\mathbb{Z}$ with $k\geq\kappa_E$ and each $\alpha,\beta\in I_k$ with $\alpha\neq\beta$ there holds $Q_\alpha^k\cap Q_\beta^k=\emptyset$;

(3) [No partial overlap across generations]
For each $k,\ell\in\mathbb{Z}$ with $\ell>k\geq\kappa_E$, and each $\alpha\in I_k$, $\beta\in I_\ell$, either $Q_\beta^\ell\subseteq Q_\alpha^k$ or $Q_\alpha^k\cap Q_\beta^\ell=\emptyset$;

(4) [Any dyadic cube has a unique ancestor in any earlier generation]
For each $k,\ell\in\mathbb{Z}$ with $k>\ell\geq\kappa_E$, and each $\alpha\in I_k$ there is a unique $\beta\in I_\ell$ such that $Q_\alpha^k\subseteq Q_\beta^\ell$;

(5) [The size is dyadically related to the generation]
For each $k\in\mathbb{Z}$ with $k\geq\kappa_E$ and each $\alpha\in I_k$ one has

$$B_\rho(x_\alpha^k, a_0 2^{-k}) \subseteq Q_\alpha^k \subseteq B_\rho(x_\alpha^k, a_1 2^{-k}); \tag{2.50}$$

In particular, given a measure σ on E for which (E,ρ,σ) is a space of homogeneous type, there exists $c>0$ such that if $Q_\beta^{k+1}\subseteq Q_\alpha^k$, then $\sigma(Q_\beta^{k+1})\geq c\sigma(Q_\alpha^k)$.

(6) [Control of the number of children]
There exists an integer $N\in\mathbb{N}$ with the property that for each $k\in\mathbb{Z}$ with $k\geq\kappa_E$ one has

$$\#\big\{\beta\in I_{k+1}:\, Q_\beta^{k+1}\subseteq Q_\alpha^k\big\}\leq N, \qquad \text{for every } \ \alpha\in I_k. \tag{2.51}$$

Furthermore, this integer may be chosen such that, for each $k\in\mathbb{Z}$ with $k\geq\kappa_E$, each $x\in E$ and $r\in(0,2^{-k})$, the number of Q's in $\mathbb{D}_k(E)$ that intersect $B_\rho(x,r)$ is at most N.

(7) [Any generation covers a dense subset of the entire space]
For each $k\in\mathbb{Z}$ with $k\geq\kappa_E$, the set $\bigcup_{\alpha\in I_k}Q_\alpha^k$ is dense in (E,τ_ρ). In particular, for each $k\in\mathbb{Z}$ with $k\geq\kappa_E$ one has

$$E=\bigcup_{\alpha\in I_k}\big\{x\in E:\, \operatorname{dist}_\rho(x,Q_\alpha^k)\leq\varepsilon 2^{-k}\big\}, \qquad \forall\,\varepsilon>0, \tag{2.52}$$

and there exist $b_0, b_1 \in (0,\infty)$ depending only on the geometrically doubling character of E with the property that

$$\begin{gathered}\forall\, x_o \in E,\ \ \forall\, r \in (0, \mathrm{diam}_\rho(E)],\ \ \exists\, k \in \mathbb{Z}\ \text{ with }\ k \geq \kappa_E\ \text{ and }\ \exists\, \alpha \in I_k \\ \text{with the property that }\ Q_\alpha^k \subseteq B_\rho(x_o, r)\ \text{ and }\ b_0 r \leq 2^{-k} \leq b_1 r.\end{gathered} \tag{2.53}$$

Moreover, for each $k \in \mathbb{Z}$ with $k \geq \kappa_E$ and each $\alpha \in I_k$

$$\bigcup_{\beta \in I_{k+1},\, Q_\beta^{k+1} \subseteq Q_\alpha^k} Q_\beta^{k+1} \quad \text{is dense in } \ Q_\alpha^k, \tag{2.54}$$

and

$$Q_\alpha^k \subseteq \bigcup_{\beta \in I_{k+1},\, Q_\beta^{k+1} \subseteq Q_\alpha^k} \big\{x \in E :\, \mathrm{dist}_\rho(x, Q_\beta^{k+1}) \leq \varepsilon 2^{-k-1}\big\}, \qquad \forall\, \varepsilon > 0. \tag{2.55}$$

(8) [Dyadic cubes have thin boundaries with respect to a background doubling measure] *Given a measure σ on E for which (E, ρ, σ) is a space of homogeneous type, a collection $\mathbb{D}(E)$ may be constructed as in* (2.49) *such that properties (1)-(7) above hold and, in addition, there exist constants $\vartheta \in (0,1)$ and $c \in (0,\infty)$ such that for each $k \in \mathbb{Z}$ with $k \geq \kappa_E$ and each $\alpha \in I_k$ one has*

$$\sigma\left(\big\{x \in Q_\alpha^k :\, \mathrm{dist}_{\rho_\#}(x, E \setminus Q_\alpha^k) \leq t\, 2^{-k}\big\}\right) \leq c\, t^\vartheta \sigma(Q_\alpha^k), \qquad \forall\, t > 0. \tag{2.56}$$

Moreover, in such a context matters may be arranged so that, for each $k \in \mathbb{Z}$ with $k \geq \kappa_E$ and each $\alpha \in I_k$,

$$\big(Q_\alpha^k, \rho|_{Q_\alpha^k}, \sigma\lfloor Q_\alpha^k\big) \qquad \text{is a space of homogeneous type,} \tag{2.57}$$

and the doubling constant of the measure $\sigma\lfloor Q_\alpha^k$ is independent of k, α (i.e., the quality of being a space of homogeneous type is hereditary at the level of dyadic cubes, in a uniform fashion*).*

(9) [All generations cover the space a.e. with respect to a doubling Borel regular measure] *If σ is a Borel measure on E which is both doubling (cf.* (2.35)*) and Borel regular (cf.* (2.28)*) then a collection $\mathbb{D}(E)$ associated with the doubling measure σ as in (8) may be constructed with the additional property that*

$$\sigma\Big(E \setminus \bigcup_{\alpha \in I_k} Q_\alpha^k\Big) = 0 \qquad \text{for each }\ k \in \mathbb{Z},\ k \geq \kappa_E. \tag{2.58}$$

In particular, in such a setting, for each $k \in \mathbb{Z}$ with $k \geq \kappa_E$ one has

$$\sigma\Big(Q_\alpha^k \setminus \bigcup_{\beta \in I_{k+1},\, Q_\beta^{k+1} \subseteq Q_\alpha^k} Q_\beta^{k+1}\Big) = 0, \qquad \text{for every }\ \alpha \in I_k. \tag{2.59}$$

Before discussing the proof of this result we wish to say a few words clarifying terminology. As already mentioned in the statement, sets Q belonging to $\mathbb{D}(E)$ will be referred to as *dyadic cubes* (on E). Also, following a well-established custom, whenever $Q_\alpha^{k+1} \subseteq Q_\beta^k$ we shall call

Q_α^{k+1} a *child* of Q_β^k, and we shall say that Q_β^k is a *parent* of Q_α^{k+1}. For a given dyadic cube, an *ancestor* is then a parent, or a parent of a parent, or so on. Moreover, for each $k \in \mathbb{Z}$ with $k \geq \kappa_E$, we shall call $\mathbb{D}_k(E)$ the *dyadic cubes of generation* k and, for each $Q \in \mathbb{D}_k(E)$, define the *side-length* of Q to be $\ell(Q) := 2^{-k}$, and the *center* of Q to be the point $x_\alpha^k \in E$ if $Q = Q_\alpha^k$.

Finally, we make the convention that saying that $\mathbb{D}(E)$ *is a dyadic cube structure (or dyadic grid) on* E will always indicate that the collection $\mathbb{D}(E)$ is associated with E as in Proposition 2.12. This presupposes that E is the ambient set for a geometrically doubling quasi-metric space, in which case $\mathbb{D}(E)$ satisfies properties *(1)-(7)* above and that, in the presence of a background measure σ satisfying appropriate conditions (as stipulated in Proposition 2.12), properties *(8)* and *(9)* also hold.

We are now ready to proceed with the

Proof of Proposition 2.12. This is a slight extension and clarification of a result proved by M. Christ in [14], generalizing earlier work by G. David in [22], and we will limit ourselves to discussing only the novel aspects of the present formulation. For the sake of reference, we debut by recalling the main steps in the construction in [14]. For a fixed real number $\delta \in (0,1)$ and for any integer k, Christ considers a maximal collection of points $z_\alpha^k \in E$ such that

$$\rho_\#(z_\alpha^k, z_\beta^k) \geq \delta^k, \qquad \forall\, \alpha \neq \beta. \tag{2.60}$$

Hence, for each fixed k, the set $\{z_\alpha^k\}_\alpha$ is δ^k-dense in E in the sense that for each $k \in \mathbb{Z}$ and $x \in E$ there exists α such that $\rho_\#(x, z_\alpha^k) < \delta^k$. Then (cf. [14, Lemma 13, p. 8]) there exists a partial order relation $\preceq$ on the set $\{(k,\alpha) : k \in \mathbb{Z}, \alpha \in I_k\}$ with the following properties:

1) if $(k,\alpha) \preceq (l,\beta)$ then $k \geq l$;
2) for each (k,α) and $l \leq k$ there exists a unique β such that $(k,\alpha) \preceq (l,\beta)$;
3) if $(k,\alpha) \preceq (k-1,\beta)$ then $\rho_\#(z_\alpha^k, z_\beta^{k-1}) < \delta^{k-1}$;
4) if $\rho_\#(z_\beta^l, z_\alpha^k) \leq 2C_\rho \delta^k$ then $(l,\beta) \preceq (k,\alpha)$.

Having established this, Christ then chooses a number $c \in (0, \frac{1}{2C_\rho})$ and defines

$$Q_\alpha^k := \bigcup_{(l,\beta) \preceq (k,\alpha)} B_{\rho_\#}(z_\beta^l, c\delta^l).$$

First, the dyadic cubes in [14, Theorem 11, p. 7] are labeled over all $k \in \mathbb{Z}$. However, (2.50) shows that in the case when E is bounded the index set I_k becomes a singleton whenever 2^{-k} is sufficiently large. Hence, in particular, $\mathbb{D}_k(E)$ becomes stationary as k approaches $-\infty$, in the sense that this collection of cubes reduces to just the set E provided 2^{-k} is sufficiently large. While this is not an issue in and of itself, for later considerations we find it useful to eliminate this redundancy and this is the reason why we restrict ourselves to only $k \geq \kappa_E$.

Second, [14, Theorem 11, p. 7] is stated with δ^k replacing 2^{-k} in (2.50)-(2.56), for some $\delta \in (0,1)$. The reason why we may always assume that $\delta = 1/2$ is discussed later below. Third, Christ's result just mentioned is formulated in the setting of spaces of homogeneous type (equipped with symmetric quasi-distance), but a cursory inspection of the proof reveals that for properties *(1)-(6)* in our statement the same type of arguments as in [14, pp. 7-10] go through (working with the regularization $\rho_\#$ of ρ, as in Theorem 2.2) under the weaker assumption that (E,ρ) is a geometrically doubling quasi-metric space.

Fourth, *(7)* follows from a careful inspection of the proof of [14, Theorem 11, p. 7], which reveals that for each $k \in \mathbb{Z}$ with $k \geq \kappa_E$, and any $j \in \mathbb{N}$ sufficiently large (compared to k)

the set $\bigcup_{\alpha\in I_k} Q^k_\alpha$ contains a 2^{-j}-dense subset of E that is maximal with respect to inclusion. Of course, this shows that the union in question is dense in (E,τ_ρ), and (2.52) is a direct consequence of it.

Fifth, with the exception of using the regularization $\rho_\#$ of the original quasi-distance ρ from Theorem 2.2 in place of the regularization devised in [56], property *(8)* is identical to condition (3.6) in [14, Theorem 11, p. 7]. Sixth, property *(9)* corresponds to (3.1) in [14, Theorem 11, p. 7] except that we are presently assuming that the doubling measure σ is Borel regular. The reason for this assumption is that the proof of (3.1) in [14, Theorem 11, p. 7] uses the Lebesgue Differentiation Theorem, whose proof requires that continuous functions vanishing outside bounded subsets of E are dense in $L^1(E,\sigma)$. It is precisely here that the aforementioned regularity of the measure intervenes and the reader is referred to [60, Theorem 7.10] for a density result of this nature.

The remainder of this proof consists of a verification that, compared with [14], it is always possible to take $\delta = 1/2$ (as described in the first paragraph of this proof). In the process, we shall adopt Christ's convention of labeling the dyadic cubes over all $k\in\mathbb{Z}$ (eliminating the inherent redundancy in the case when E is bounded may be done afterwards). To get started, let $\mathfrak{D}(E) := \bigcup\limits_{k\in\mathbb{Z}} \mathfrak{D}_k(E)$ denote a collection of dyadic cubes enjoying properties *(1)-(9)* listed in Proposition 2.12 but with δ^k replacing 2^{-k} in (2.50)-(2.56). Our goal in this part of the proof is to construct another collection of dyadic cubes, $\mathbb{D}(E) := \bigcup\limits_{k\in\mathbb{Z}} \mathbb{D}_k(E)$ satisfying similar properties for $\delta = 1/2$. We shall consider two cases.

Case I: $1/2 < \delta < 1$. Set $m_0 := 0$ and, for each integer $k > 0$, let m_k be the largest positive integer such that $\delta^{m_k} \geq 2^{-k}$. Thus,

$$\delta^{m_k+1} < 2^{-k} \leq \delta^{m_k}. \tag{2.61}$$

Similarly, for each $k < 0$, let m_k denote the least integer such that $\delta^{m_k+1} < 2^{-k}$. Thus, again we have (2.61). Of course, we shall have $m_k < 0$ when $k < 0$. The sequence $\{m_k\}_{k\in\mathbb{Z}}$ is strictly increasing. Indeed, for every $k\in\mathbb{Z}$, we have

$$m_k + 1 \leq m_{k+1}. \tag{2.62}$$

To see this in the case that $k \geq 0$, observe that

$$2^{-k-1} = \tfrac{1}{2}2^{-k} \leq \tfrac{1}{2}\delta^{m_k} < \delta^{m_k+1}, \tag{2.63}$$

where in the first inequality we have used (2.61) and in the second that $1/2 < \delta$. Thus, (2.62) holds, since by definition m_{k+1} is the greatest integer for which $2^{-(k+1)} \leq \delta^{m_{k+1}}$. In the case $k \leq 0$, since $1 < 2\delta$ we have

$$\delta^{(m_{k+1}-1)+1} < 2\delta^{m_{k+1}+1} < 2\cdot 2^{-(k+1)} = 2^{-k} \tag{2.64}$$

where in the second inequality we have used (2.61). Since m_k is the smallest integer for which $\delta^{m_k+1} < 2^{-k}$, we again obtain (2.62).

We then define

$$\mathbb{D}_k(E) := \mathfrak{D}_{m_k}(E). \tag{2.65}$$

It is routine to verify that $\mathbb{D}_k(E)$ satisfies the desired properties, with some of the constants possibly depending upon δ.

Case II: $0 < \delta < 1/2$. In this case we reverse the roles of $1/2$ and δ in the construction in Case I above, to construct a strictly increasing sequence of integers $\{m_k\}_{k\in\mathbb{Z}}$, with $m_0 := 0$, for which

$$2^{-m_k} \leq \delta^k < 2^{-m_k+1}, \qquad \forall\, k \in \mathbb{Z}. \tag{2.66}$$

It then follows that there is a fixed positive integer $q_0 \approx \log_2(1/\delta)$ such that for each $k \in \mathbb{Z}$,

$$m_{k+1} - q_0 \leq m_k < m_{k+1}. \tag{2.67}$$

Indeed, we have

$$2^{-m_k} \leq \delta^k = \frac{1}{\delta}\delta^{k+1} < \frac{1}{\delta}2^{-m_{k+1}+1} = \frac{2}{\delta}2^{-m_{k+1}}, \tag{2.68}$$

where in the two inequalities we have used (2.66). We then obtain (2.67) by taking logarithms. For each $k \in \mathbb{Z}$ we now set

$$\mathbb{D}_j(E) := \mathfrak{D}_k(E), \qquad m_k \leq j < m_{k+1}. \tag{2.69}$$

It is now routine to check that the collection $\mathbb{D}(E) := \bigcup_{k\in\mathbb{Z}} \mathbb{D}_k(E)$, so defined, satisfies the desired properties, with some of the constants possibly depending on δ. In verifying the various properties, it is helpful to observe that by (2.67), we have that

$$2^{-j} \approx 2^{-m_k} \approx \delta^k, \qquad \text{whenever } m_k \leq j < m_{k+1}. \tag{2.70}$$

This finishes the proof of the proposition. □

2.3 Approximations to the identity on quasi-metric spaces

This subsection is devoted to reviewing the definition and properties of approximations to the identity on ADR spaces. To set the stage, we make the following definition.

Definition 2.13. *Assume that (E,ρ,σ) is a d-dimensional* ADR *space for some $d > 0$ and recall $\kappa_E \in \mathbb{Z} \cup \{-\infty\}$ from* (2.48)*. In this context, call a family $\{\mathcal{S}_l\}_{l\in\mathbb{Z},\, l\geq\kappa_E}$ of integral operators*

$$\mathcal{S}_l f(x) := \int_E S_l(x,y) f(y)\, d\sigma(y), \qquad x \in E, \tag{2.71}$$

with integral kernels $S_l : E \times E \to \mathbb{R}$, an `approximation to the identity of order` γ *on E provided there exists a finite constant $C > 0$ such that, for every $l \in \mathbb{Z}$ with $l \geq \kappa_E$, the following properties hold:*

(i) $0 \leq S_l(x,y) \leq C2^{ld}$ *for all* $x, y \in E$*, and* $S_l(x,y) = 0$ *if* $\rho(x,y) \geq C2^{-l}$*;*

(ii) $|S_l(x,y) - S_l(x',y)| \leq C2^{l(d+\gamma)}\rho(x,x')^\gamma$ *for every* $x, x', y \in E$*;*

(iii) $\big|[S_l(x,y) - S_l(x',y)] - [S_l(x,y') - S_l(x',y')]\big| \leq C2^{l(d+2\gamma)}\rho(x,x')^\gamma\rho(y,y')^\gamma$ *for every point* $x, x', y, y' \in E$*;*

(iv) $S_l(x,y) = S_l(y,x)$ *for every* $x, y \in E$*, and* $\int_E S_l(x,y)\, d\sigma(y) = 1$ *for every* $x \in E$*.*

Starting with the work of Coifman (cf. the discussion in [24, pp. 16-17 and p. 40]), the existence of approximations to the identity of some order $\gamma > 0$ on ADR spaces has been established in [24, p. 40], [35, pp. 10-11], [27, p. 16] (at least when $d = 1$) for various values of $\gamma > 0$ and, more recently, in [60] for the value of the order parameter γ which is optimal in relation to the quasi-metric space structure. From [60], we quote the following result:

Proposition 2.14. *Let (E, ρ, σ) be a d-dimensional* ADR *space for some $d > 0$ and assume that*

$$0 < \gamma < \min\{d + 1, \alpha_\rho\}, \tag{2.72}$$

where the index $\alpha_\rho \in (0, \infty]$ is associated to the quasi-distance ρ as in (2.11)*. Then, in the sense of Definition 2.13, there exists an approximation to the identity of order γ on E, denoted by $\{\mathcal{S}_l\}_{l \in \mathbb{Z},\, l \geq \kappa_E}$. Furthermore, given $p \in (1, \infty)$ and $f \in L^p(E, \sigma)$, it follows that:*

$$\sup_{l \in \mathbb{Z},\, l \geq \kappa_E} \big\|\mathcal{S}_l\big\|_{L^p(E,\sigma) \to L^p(E,\sigma)} < +\infty, \tag{2.73}$$

$$\text{if the measure } \sigma \text{ is Borel regular on } (E, \tau_\rho) \implies \lim_{l \to +\infty} \mathcal{S}_l f = f \quad \text{in } L^p(E, \sigma), \tag{2.74}$$

and

$$\text{if } \operatorname{diam}_\rho(E) = +\infty \implies \lim_{l \to -\infty} \mathcal{S}_l f = 0 \quad \text{in } L^p(E, \sigma). \tag{2.75}$$

Later on we shall need a Calderón-type reproducing formula involving the conditional expectation operators associated with an approximation to the identity, as discussed above. While this is a topic treated at some length in [24], [27], [35], we prove below a version of this result which best suits the purposes we have in mind.

To state the result, we first record the following preliminaries.

Definition 2.15. *A series $\sum\limits_{j \in \mathbb{N}} x_j$ of vectors in a Banach space $\mathscr{B}$ is said to be* `unconditionally convergent` *if the series $\sum\limits_{j=1}^{\infty} x_{\sigma(j)}$ converges in $\mathscr{B}$ for all permutations σ of $\mathbb{N}$.*

Clearly, if a series $\sum_{j \in \mathbb{N}} x_j$ of vectors in a Banach space $\mathscr{B}$ is unconditionally convergent then so is $\sum\limits_{j=1}^{\infty} x_{\sigma(j)}$ for any permutation σ of $\mathbb{N}$. It is also well-known (cf., e.g., [36, Corollary 3.11, p. 99]) that, given a sequence of vectors $\{x_j\}_{j \in \mathbb{N}}$ in a Banach space $\mathscr{B}$,

$$\begin{aligned} &\sum_{j \in \mathbb{N}} x_j \text{ unconditionally convergent} \\ &\qquad \implies \sum_{j=1}^{\infty} x_{\sigma_1(j)} = \sum_{j=1}^{\infty} x_{\sigma_2(j)}, \qquad \forall\, \sigma_1, \sigma_2 \text{ permutations of } \mathbb{N}. \end{aligned} \tag{2.76}$$

Hence, whenever $\sum\limits_{j \in \mathbb{N}} x_j$ is unconditionally convergent, we may unambiguously define

$$\sum_{j \in \mathbb{N}} x_j := \sum_{j=1}^{\infty} x_{\sigma(j)} \quad \text{for some (hence any) permutation } \sigma \text{ of } \mathbb{N}. \tag{2.77}$$

Let us also record here the following useful characterizations of unconditional convergence (in a Banach space setting):

$$\begin{aligned}\sum_{j\in\mathbb{N}} x_j \text{ unconditionally convergent } &\iff \sum_{j=1}^{\infty} \varepsilon_j x_j \text{ convergent } \ \forall\, \varepsilon_j = \pm 1 \\ &\iff \begin{cases} \forall\, \varepsilon > 0\ \exists\, N_\varepsilon \in \mathbb{N} \text{ such that } \Big\| \sum_{j\in\mathcal{I}} x_j \Big\| < \varepsilon \\ \forall\, \mathcal{I} \text{ finite subset of } \mathbb{N} \text{ with } \min \mathcal{I} \geq N_\varepsilon. \end{cases}\end{aligned} \tag{2.78}$$

See, e.g., [36, Theorem 3.10, p. 94] where these and other equivalent characterizations are proved.

The following notion of unconditional convergence applies to series indexed by any countable set other than $\mathbb{N}$.

Definition 2.16. *For any countable set $\mathbb{I}$, a series $\sum_{j\in\mathbb{I}} x_j$ of vectors in a Banach space $\mathscr{B}$ is said to be* `unconditionally convergent` *if there exists a bijection $\varphi : \mathbb{N} \to \mathbb{I}$ such that $\sum_{j\in\mathbb{N}} x_{\varphi(j)}$ is unconditionally convergent in the sense of Definition 2.15, in which case the sum of the series in $\mathscr{B}$ is defined as $\sum_{j\in\mathbb{I}} x_j := \sum\limits_{j=1}^{\infty} x_{\varphi(j)}$.*

Note that the property of being unconditionally convergent as introduced in Definition 2.16 is independent of the bijection φ used. To see this, suppose that $\sum_{j\in\mathbb{I}} x_j$ is unconditionally convergent in $\mathscr{B}$ and let $\varphi : \mathbb{N} \to \mathbb{I}$ be a bijection such that $\sum_{j\in\mathbb{N}} x_{\varphi(j)}$ is unconditionally convergent in the sense of Definition 2.15. If $\widetilde{\varphi} : \mathbb{N} \to \mathbb{I}$ is another bijection, then $\varphi^{-1} \circ \widetilde{\varphi}$ is a permutation of $\mathbb{N}$ hence, as noted right after Definition 2.15, $\sum_{j\in\mathbb{N}} x_{\widetilde{\varphi}(j)}$ is also unconditionally convergent. Moreover, (2.76) ensures that $\sum\limits_{j=1}^{\infty} x_{\widetilde{\varphi}(j)} = \sum\limits_{j=1}^{\infty} x_{\varphi(\varphi^{-1}(\widetilde{\varphi}(j)))} = \sum\limits_{j=1}^{\infty} x_{\varphi(j)} = \sum_{j\in\mathbb{I}} x_j$.

We also have the following equivalent characterization for unconditional convergence.

Lemma 2.17. *Suppose $\mathscr{B}$ is a Banach space and $\mathbb{I}$ is a countable set. Then a series $\sum_{j\in\mathbb{I}} x_j$ of vectors in $\mathscr{B}$ is unconditionally convergent in $\mathscr{B}$ if and only if*

$$\begin{gathered}\forall\, \{S_i\}_{i\in\mathbb{N}} \text{ such that } S_i \text{ finite and } S_i \subseteq S_{i+1} \subseteq \mathbb{I} \text{ for each } i \in \mathbb{N}, \\ \text{the sequence } \Big\{ \sum_{j\in S_i} x_j \Big\}_{i\in\mathbb{N}} \text{ converges in } \mathscr{B}.\end{gathered} \tag{2.79}$$

Proof. Suppose $\sum_{j\in\mathbb{I}} x_j$ is such that (2.79) holds and let $\varphi : \mathbb{N} \to \mathbb{I}$ be a bijection. Also fix an arbitrary permutation $\sigma : \mathbb{N} \to \mathbb{N}$. Then the sequence $S_i := \{\varphi(\sigma(k)) : 1 \leq k \leq i\}$, $i \in \mathbb{N}$, of subsets of $\mathbb{I}$ satisfies the conditions in the first line of (2.79). Hence, $\big\{ \sum\limits_{j=1}^{i} x_{\varphi(\sigma(j))} \big\}_{i\in\mathbb{N}}$ is convergent in $\mathscr{B}$, which is equivalent with $\sum\limits_{j=1}^{\infty} x_{\varphi(\sigma(j))}$ being convergent in $\mathscr{B}$. Since the permutation σ of $\mathbb{N}$ has been arbitrarily chosen, this shows that $\sum_{j\in\mathbb{N}} x_{\varphi(j)}$ is unconditionally convergent in $\mathscr{B}$, thus $\sum_{j\in\mathbb{I}} x_j$ is unconditionally convergent in $\mathscr{B}$. For the converse implication, suppose that $\sum_{j\in\mathbb{I}} x_j$ is unconditionally convergent in $\mathscr{B}$. Thus, for any bijection $\varphi : \mathbb{N} \to \mathbb{I}$ we have that $\sum_{j\in\mathbb{N}} x_{\varphi(j)}$ is unconditionally convergent in $\mathscr{B}$. Let $\{S_i\}_{i\in\mathbb{N}}$ be as in the first line of (2.79) and set $S := \bigcup\limits_{i\in\mathbb{N}} S_i$. Using (2.78), it follows that $\sum\limits_{j\in\mathbb{N}\setminus\varphi^{-1}(\mathbb{I}\setminus S)} x_{\varphi(j)}$ is also unconditionally

convergent in $\mathscr{B}$. In turn, the latter readily implies that $\big\{\sum_{j\in S_i} x_j\big\}_{i\in\mathbb{N}}$ is convergent in $\mathscr{B}$, as wanted. $\square$

We now state the aforementioned Calderón-type reproducing formula.

Proposition 2.18. *Let (E,ρ,σ) be a d-dimensional* ADR *space for some $d>0$ and assume that the measure σ is Borel regular on (E,τ_ρ). In this context, recall κ_E from* (2.48) *and, for some fixed γ as in* (2.72)*, let $\{\mathcal{S}_l\}_{l\in\mathbb{Z},\,l\geq\kappa_E}$ be an approximation to the identity of order γ on E (cf. Proposition 2.14). Finally, introduce the integral operators (see [24])*

$$D_l := \mathcal{S}_{l+1} - \mathcal{S}_l, \quad l\in\mathbb{Z},\ l\geq\kappa_E. \tag{2.80}$$

Then there exist a linear and bounded operator R on $L^2(E,\sigma)$ and a family $\big\{\widetilde{D}_l\big\}_{l\in\mathbb{Z},\,l\geq\kappa_E}$ of linear operators on $L^2(E,\sigma)$ with the property that

$$\sum_{l\in\mathbb{Z},\,l\geq\kappa_E} \|\widetilde{D}_l f\|^2_{L^2(E,\sigma)} \leq C\|f\|^2_{L^2(E,\sigma)}, \qquad \textit{for each}\ \ f\in L^2(E,\sigma), \tag{2.81}$$

and, with I denoting the identity operator on $L^2(E,\sigma)$,

$$I + \mathcal{S}_{\kappa_E} R = \sum_{l\in\mathbb{Z},\,l\geq\kappa_E} D_l\widetilde{D}_l \qquad \textit{pointwise unconditionally in } L^2(E,\sigma), \tag{2.82}$$

with the convention (taking effect when $\operatorname{diam}_\rho(E)=+\infty$*) that $\mathcal{S}_{-\infty}:=0$.*

As a preamble to the proof of the above proposition we momentarily digress and record a version of the Cotlar-Knapp-Stein lemma which suits our purposes.

Lemma 2.19. *Assume that $\mathscr{H}_0$, $\mathscr{H}_1$ are two Hilbert spaces and consider a family of operators $\{T_j\}_{j\in\mathbb{I}}$, indexed by a countable set $\mathbb{I}$, with $T_j:\mathscr{H}_0\to\mathscr{H}_1$ linear and bounded for every $j\in\mathbb{I}$. Then, if the T_j's are almost orthogonal in the sense that*

$$C_0 := \sup_{j\in\mathbb{I}}\Big(\sum_{k\in\mathbb{I}}\sqrt{\|T_j^*T_k\|_{\mathscr{H}_0\to\mathscr{H}_0}}\Big)<\infty, \quad C_1 := \sup_{k\in\mathbb{I}}\Big(\sum_{j\in\mathbb{I}}\sqrt{\|T_jT_k^*\|_{\mathscr{H}_1\to\mathscr{H}_1}}\Big)<\infty \tag{2.83}$$

it follows that for any subset J of $\mathbb{I}$,

$$\begin{gathered}\sum_{j\in J} T_j x \ \textit{ converges unconditionally in } \mathscr{H}_1 \textit{ for each } x\in\mathscr{H}_0, \textit{ and}\\ \textit{if } \Big(\sum_{j\in J} T_j\Big)x := \sum_{j\in J} T_j x \ \textit{ then } \ \Big\|\sum_{j\in J} T_j\Big\|_{\mathscr{H}_0\to\mathscr{H}_1} \leq \sqrt{C_0C_1}.\end{gathered} \tag{2.84}$$

Furthermore,

$$\Big(\sum_{j\in\mathbb{I}}\|T_jx\|^2_{\mathscr{H}_1}\Big)^{1/2} \leq 2\sqrt{C_0C_1}\,\|x\|_{\mathscr{H}_0}, \qquad \forall\, x\in\mathscr{H}_0. \tag{2.85}$$

Proof. This result is typically stated with J finite and without including (2.85). See, for example, [73, Lemma 4.1, p. 285] as well as [71, Theorem 1, p.280 and comment following it]. The fact that the more general version formulated above holds is an immediate consequence of the standard version of the Cotlar-Knapp-Stein lemma as stated in the aforementioned references and the abstract functional analytic result contained in Lemma 2.20 below. $\square$

Lemma 2.20. *Let $\mathscr{H}$ be a Hilbert space with norm $\|\cdot\|_{\mathscr{H}}$ and assume that $\{x_j\}_{j\in\mathbb{I}}$ is a sequence of vectors in $\mathscr{H}$ indexed by a countable set $\mathbb{I}$. Then*

$$\Big(\sum_{j\in\mathbb{I}}\|x_j\|_{\mathscr{H}}^2\Big)^{1/2} \leq 2\cdot \sup_{\substack{J_o\subseteq\mathbb{I}\\ J_o\,\textit{finite}}}\Big\|\sum_{j\in J_o}x_j\Big\|_{\mathscr{H}}, \tag{2.86}$$

and

$$\sum_{j\in\mathbb{I}}x_j \text{ is unconditionally convergent} \Longleftrightarrow \sup_{\substack{J_o\subseteq\mathbb{I}\\ J_o\,\textit{finite}}}\Big\|\sum_{j\in J_o}x_j\Big\|_{\mathscr{H}}<\infty. \tag{2.87}$$

Moreover, if the above supremum is finite, then

$$\Big\|\sum_{j\in\mathbb{I}}x_j\Big\|_{\mathscr{H}} \leq \sup_{\substack{J_o\subseteq\mathbb{I}\\ J_o\,\textit{finite}}}\Big\|\sum_{j\in J_o}x_j\Big\|_{\mathscr{H}}. \tag{2.88}$$

Proof. It suffices to assume that $\mathbb{I}=\mathbb{N}$. This follows from Definition 2.16, since for any bijection $\varphi:\mathbb{N}\to\mathbb{I}$, we have

$$\sup_{\substack{J_o\subseteq\mathbb{N}\\ J_o\,\text{finite}}}\Big\|\sum_{n\in J_o}x_{\varphi(n)}\Big\|_{\mathscr{H}} = \sup_{\substack{J_o\subseteq\mathbb{I}\\ J_o\,\text{finite}}}\Big\|\sum_{j\in J_o}x_j\Big\|_{\mathscr{H}}. \tag{2.89}$$

We begin by establishing (2.86). To get started, let $\{x_j\}_{j\in\mathbb{N}}\subseteq\mathscr{H}$ be such that

$$C:=\sup_{\substack{J_o\subseteq\mathbb{N}\\ J_o\,\text{finite}}}\Big\|\sum_{j\in J_o}x_j\Big\|_{\mathscr{H}}<\infty. \tag{2.90}$$

Assume that $\{r_j\}_{j\in\mathbb{N}}$ is Rademacher's system of functions on $[0,1]$, i.e., for each $j\in\mathbb{N}$,

$$r_j(t)=\operatorname{sign}\big(\sin(2^j\pi t)\big)\in\{-1,0,+1\},\qquad \text{for all } t\in[0,1]. \tag{2.91}$$

Hence, in particular,

$$\int_0^1 r_j(t)r_k(t)\,dt=\delta_{jk},\qquad \forall\, j,k\in\mathbb{N}. \tag{2.92}$$

Consequently, if $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ stands for the inner product in $\mathscr{H}$, then for any finite set $J_o\subseteq\mathbb{N}$,

$$\begin{aligned}\int_0^1\Big\|\sum_{j\in J_o}r_j(t)x_j\Big\|_{\mathscr{H}}^2dt &= \int_0^1\Big\langle\sum_{j\in J_o}r_j(t)x_j,\sum_{k\in J_o}r_k(t)x_k\Big\rangle_{\mathscr{H}}dt \qquad (2.93)\\ &= \sum_{j,k\in J_o}\Big(\int_0^1 r_j(t)r_k(t)\,dt\Big)\langle x_j,x_k\rangle_{\mathscr{H}}=\sum_{j\in J_o}\|x_j\|_{\mathscr{H}}^2.\end{aligned}$$

On the other hand, thanks to (2.91), for each $t\in[0,1]$ we may estimate

$$\begin{aligned}\Big\|\sum_{j\in J_o}r_j(t)x_j\Big\|_{\mathscr{H}} &= \Big\|\Big(\sum_{j\in J_o,\,r_j(t)=+1}x_j\Big)-\Big(\sum_{j\in J_o,\,r_j(t)=-1}x_j\Big)\Big\|_{\mathscr{H}}\\ &\leq \Big\|\sum_{j\in J_o,\,r_j(t)=+1}x_j\Big\|_{\mathscr{H}}+\Big\|\sum_{j\in J_o,\,r_j(t)=-1}x_j\Big\|_{\mathscr{H}}\leq 2C. \qquad (2.94)\end{aligned}$$

By combining (2.93) and (2.94) we therefore obtain

$$\sum_{j\in J_o}\|x_j\|_{\mathscr{H}}^2 \le 4C^2, \qquad \text{for every finite subset } J_o \text{ of } \mathbb{N}, \tag{2.95}$$

from which (2.86) readily follows.

Moving on, assume that (2.90) holds yet $\sum_{j\in\mathbb{N}} x_j$ does not converge unconditionally, and seek a contradiction. Then (cf. the first equivalence in (2.78)), there exists a choice of signs $\varepsilon_j \in \{\pm 1\}$, $j \in \mathbb{N}$, with the property that the sequence of partial sums of the series $\sum_{j\in\mathbb{N}} \varepsilon_j x_j$ is not Cauchy in $\mathscr{H}$. In turn, this implies that there exist $\vartheta > 0$ along with two sequences $\{a_i\}_{i\in\mathbb{N}}$, $\{b_i\}_{i\in\mathbb{N}}$ of numbers in $\mathbb{N}$, such that

$$a_i \le b_i < a_{i+1} \quad \text{ and } \quad \Big\|\sum_{a_i\le j\le b_i} \varepsilon_j x_j\Big\|_{\mathscr{H}} \ge \vartheta, \qquad \text{for every } i \in \mathbb{N}. \tag{2.96}$$

In this scenario, consider the sequence $\{y_i\}_{i\in\mathbb{N}}$ of vectors in $\mathscr{H}$ defined by

$$y_i := \sum_{a_i\le j\le b_i} \varepsilon_j x_j \ \text{ for every } i \in \mathbb{N}, \tag{2.97}$$

and note that, by (2.96),

$$\|y_i\|_{\mathscr{H}} \ge \vartheta, \qquad \text{for every } i \in \mathbb{N}. \tag{2.98}$$

Fix now an arbitrary finite subset I_o of $\mathbb{N}$ and set $J_o := \big\{j \in \mathbb{N} : \exists\, i \in I_o \text{ such that } a_i \le j \le b_i\big\}$. Thus, J_o is a finite subset of $\mathbb{N}$. Then with the constant C as in (2.90), we have

$$\begin{aligned}\Big\|\sum_{i\in I_o} y_i\Big\|_{\mathscr{H}} &= \Big\|\sum_{i\in I_o}\Big(\sum_{a_i\le j\le b_i}\varepsilon_j x_j\Big)\Big\|_{\mathscr{H}} = \Big\|\Big(\sum_{j\in J_o,\,\varepsilon_j=+1} x_j\Big) - \Big(\sum_{j\in J_o,\,\varepsilon_j=-1} x_j\Big)\Big\|_{\mathscr{H}} \\ &\le \Big\|\sum_{j\in J_o,\,\varepsilon_j=+1} x_j\Big\|_{\mathscr{H}} + \Big\|\sum_{j\in J_o,\,\varepsilon_j=-1} x_j\Big\|_{\mathscr{H}} \le 2C,\end{aligned} \tag{2.99}$$

where the second equality relies on the fact from (2.96) that $a_i \le b_i < a_{i+1}$. Hence,

$$\sup_{\substack{I_o\subseteq\mathbb{N}\\ I_o \text{ finite}}}\Big\|\sum_{i\in I_o} y_i\Big\|_{\mathscr{H}} \le 2C. \tag{2.100}$$

Having established this, (2.86) then gives $\sum_{i\in\mathbb{N}}\|y_i\|_{\mathscr{H}}^2 \le 16C^2 < \infty$ which, in particular, forces $\lim\limits_{i\to\infty}\|y_i\|_{\mathscr{H}} = 0$. This, however, contradicts (2.98).

To summarize, the proof so far shows that if (2.90) holds then the series $\sum_{j\in\mathbb{N}} x_j$ is unconditionally convergent. Of course, once the (norm) convergence of the series has been established then (2.90) also gives $\Big\|\sum\limits_{j\in\mathbb{N}} x_j\Big\|_{\mathscr{H}} \le \limsup\limits_{N\to\infty}\Big\|\sum\limits_{j=1}^{N} x_j\Big\|_{\mathscr{H}} \le C$, proving (2.88).

There remains to prove that the finiteness condition in (2.90) holds if the series $\sum_{j\in\mathbb{N}} x_j$ is unconditionally convergent. With $N_1 \in \mathbb{N}$ denoting the integer N_ε corresponding to taking $\varepsilon = 1$ in the last condition in (2.78), consider

$$M := \sup_{I_o\subseteq\{1,\dots,N_1\}}\Big\|\sum_{j\in I_o} x_j\Big\|_{\mathscr{H}} < \infty. \tag{2.101}$$

Then, given any finite subset J_o of $\mathbb{N}$ we may write

$$\Big\|\sum_{j\in J_o} x_j\Big\|_{\mathscr{H}} \leq \Big\|\sum_{j\in J_o\cap\{1,\dots,N_1\}} x_j\Big\|_{\mathscr{H}} + \Big\|\sum_{j\in J_o\setminus\{1,\dots,N_1\}} x_j\Big\|_{\mathscr{H}} \leq M+1, \tag{2.102}$$

from which the finiteness condition in (2.90) follows. □

For further reference, given an ambient quasi-metric space $(\mathscr{X},\rho)$ and a set E with the property that there exists a Borel measure σ on $(E,\tau_{\rho|_E})$ such that $(E,\rho_\#|_E,\sigma)$ is a space of homogeneous type, we shall denote by M_E the `Hardy-Littlewood maximal function` in this context, i.e.,

$$(M_E f)(x) := \sup_{r>0} \frac{1}{\sigma\big(B_{\rho_\#}(x,r)\big)} \int_{B_{\rho_\#}(x,r)} |f(y)|\,d\sigma(y), \qquad x\in E. \tag{2.103}$$

We next present the

Proof of Proposition 2.18. For each $l\in\mathbb{Z}$ with $l\geq\kappa_E$, denote by $h_l(\cdot,\cdot)$ the integral kernel of the operator D_l. Thus, $h_l(\cdot,\cdot)=S_{l+1}(\cdot,\cdot)-S_l(\cdot,\cdot)$ and, as a consequence of properties $(i)-(iv)$ in Definition 2.13, we see that $h_l(\cdot,\cdot)$ is a symmetric function on $E\times E$, and there exists $C\in(0,\infty)$ such that for each $l\in\mathbb{Z}$ with $l\geq\kappa_E$ we have

$$|h_l(\cdot,\cdot)| \leq C2^{ld}\mathbf{1}_{\{\rho(\cdot,\cdot)\leq C2^{-l}\}}, \qquad \text{on } \ E\times E, \tag{2.104}$$

$$|h_l(x,y)-h_l(x',y)| \leq C2^{l(d+\gamma)}\rho(x,x')^\gamma \quad \forall\, x,x',y\in E, \tag{2.105}$$

$$\int_E h_l(x,y)\,d\sigma(x) = 0 \quad \forall\, y\in E. \tag{2.106}$$

Of course, due to the symmetry of h, smoothness and cancellation conditions in the second variable, similar to (2.105) and (2.106), respectively, also hold.

Furthermore, for each $j,k\in\mathbb{Z}$ with $j,k\geq\kappa_E$, using first (2.106), then (2.104) and (2.105), and then the fact that (E,ρ,σ) is d-ADR, we may write

$$\begin{aligned} \Big|\int_E h_j(x,z)h_k(z,y)\,d\sigma(z)\Big| &= \Big|\int_E [h_j(x,z)-h_j(x,y)]h_k(z,y)\,d\sigma(z)\Big| \\ &\leq C2^{j(d+\gamma)}\int_E \rho_\#(y,z)^\gamma 2^{kd}\mathbf{1}_{\{\rho_\#(y,\cdot)\leq C2^{-k}\}}(z)\,d\sigma(z) \\ &\leq C2^{j(d+\gamma)}2^{-k\gamma}. \end{aligned} \tag{2.107}$$

Similarly,

$$\begin{aligned} \Big|\int_E h_j(x,z)h_k(z,y)\,d\sigma(z)\Big| &= \Big|\int_E h_j(x,z)[h_k(z,y)-h_k(x,y)]\,d\sigma(z)\Big| \\ &\leq C2^{k(d+\gamma)}2^{-j\gamma}. \end{aligned} \tag{2.108}$$

Combining (2.107), (2.108), and the support condition (2.104), it follows that for each $j,k\in\mathbb{Z}$ with $j,k\geq\kappa_E$, there holds (compare with [24, p. 15] and [27, (1.14), p. 16])

$$\Big|\int_E h_j(x,z)h_k(z,y)\,d\sigma(z)\Big| \leq C2^{-|j-k|\gamma}\,2^{d\cdot\min(j,k)}\mathbf{1}_{\{\rho(x,y)\leq C2^{-\min(j,k)}\}}, \quad \forall\, x,y\in E. \tag{2.109}$$

Note that for each $j,k\in\mathbb{Z}$ with $j,k\geq\kappa_E$ we have that $D_jD_k: L^2(E,\sigma)\to L^2(E,\sigma)$ is a linear and bounded integral operator whose integral kernel is given by $\int_E h_j(x,z)h_k(z,y)\,d\sigma(z)$, for $x,y\in E$. Based on this and (2.109) we may then conclude that for each $j,k\in\mathbb{Z}$ with $j,k\geq\kappa_E$,

$$
\begin{aligned}
\big|(D_jD_kf)(x)\big| &\leq C2^{-|j-k|\gamma}\fint_{B_{\rho_\#}(x,C2^{-\min(j,k)})}|f(y)|\,d\sigma(y)\\
&\leq C2^{-|j-k|\gamma}M_E(f)(x),\qquad\forall\,x\in E,
\end{aligned}
\tag{2.110}
$$

for every $f\in L^1_{loc}(E,\sigma)$. In turn, the boundedness of M_E and (2.110) yield

$$
\|D_jD_k\|_{L^2(E,\sigma)\to L^2(E,\sigma)}\leq C2^{-|j-k|\gamma},\qquad\forall\,j,k\in\mathbb{Z},\ j,k\geq\kappa_E. \tag{2.111}
$$

Having established (2.111), it follows that the family of linear operators $\big\{D_l\big\}_{l\in\mathbb{Z},\,l\geq\kappa_E}$, from $L^2(E,\sigma)$ into itself, is almost orthogonal. As such, Lemma 2.19 applies and gives that

$$
\sup_{\substack{J\subseteq\mathbb{Z}\\ J\text{ finite}}}\Big\|\sum_{l\in J,\,l\geq\kappa_E}D_l\Big\|_{L^2(E,\sigma)\to L^2(E,\sigma)}\leq C<\infty, \tag{2.112}
$$

the following Littlewood-Paley estimate holds

$$
\Big(\sum_{l\in\mathbb{Z},\,l\geq\kappa_E}\|D_lf\|^2_{L^2(E,\sigma)}\Big)^{1/2}\leq C\|f\|_{L^2(E,\sigma)},\qquad\text{for each}\quad f\in L^2(E,\sigma), \tag{2.113}
$$

and, making use of (2.74) and (2.75) as well, we have

$$
\begin{gathered}
\big(I-\mathcal{S}_{\kappa_E}\big)f=\sum_{l\in\mathbb{Z},\,l\geq\kappa_E}D_lf\qquad\text{for each } f\in L^2(E,\sigma),\\
\text{where the series converges unconditionally in } L^2(E,\sigma).
\end{gathered}
\tag{2.114}
$$

To proceed, fix a number $N\in\mathbb{N}$. Based on (2.112), we may square (2.114) and obtain, pointwise in $L^2(E,\sigma)$,

$$
\begin{aligned}
\big(I-\mathcal{S}_{\kappa_E}\big)^2 &= \lim_{M\to\infty}\Big[\Big(\sum_{j\in\mathbb{Z},\,j\geq\kappa_E,\,|j|\leq M}D_j\Big)\Big(\sum_{k\in\mathbb{Z},\,k\geq\kappa_E,\,|k|\leq M}D_k\Big)\Big]\\
&= \lim_{M\to\infty}\Big(\sum_{\substack{|j-k|\leq N\\ j,k\geq\kappa_E,\,|j|,|k|\leq M}}D_jD_k+\sum_{\substack{|j-k|>N\\ j,k\geq\kappa_E,\,|j|,|k|\leq M}}D_jD_k\Big).
\end{aligned}
\tag{2.115}
$$

Going further, fix $i\in\mathbb{Z}$ and consider the family $\{T_l\}_{l\in J_i}$ of operators on $L^2(E,\sigma)$, where

$$
T_l:=D_{l+i}D_l\qquad\text{for every}\qquad l\in J_i:=\big\{l\in\mathbb{Z}:\ l\geq\max\{\kappa_E,\kappa_E-i\}\big\}. \tag{2.116}
$$

Then, with $\|\cdot\|$ temporarily abbreviating $\|\cdot\|_{L^2(E,\sigma)\to L^2(E,\sigma)}$, for each $j,k\in J_i$ we may estimate

$$
\begin{aligned}
\|T_j^*T_k\| &\leq \min\Big\{\|D_j\|\|D_{j+i}D_{k+i}\|\|D_k\|\,,\ \|D_jD_{j+i}\|\|D_{k+i}\|\|D_k\|\Big\}\\
&\leq C\,\min\Big\{2^{-|k-j|\gamma}\,,\ 2^{-|i|\gamma}\Big\},
\end{aligned}
\tag{2.117}
$$

thanks to (2.112) and (2.111). This readily implies that $\sup_{j\in J_i}\Big(\sum_{k\in J_i}\sqrt{\|T_j^*T_k\|}\Big)\leq C(1+|i|)2^{-|i|\gamma/2}$ and $\sup_{k\in J_i}\Big(\sum_{j\in J_i}\sqrt{\|T_j\,T_k^*\|}\Big)\leq C(1+|i|)2^{-|i|\gamma/2}$ for some $C\in(0,\infty)$ independent of i. Hence, for each $i\in\mathbb{Z}$, the family $\big\{D_{l+i}D_l\big\}_{l\in\mathbb{Z},\,l\geq\max\{\kappa_E,\kappa_E-i\}}$ is almost orthogonal, and by Lemma 2.19 there exists some constant $C\in(0,\infty)$ independent of i such that for every set $J\subseteq J_i$ we have that $\sum_{l\in J}D_{l+i}D_l$ converges pointwise unconditionally in $L^2(E,\sigma)$ and

$$\Big\|\sum_{l\in J}D_{l+i}D_l\Big\|_{L^2(E,\sigma)\to L^2(E,\sigma)}\leq C(1+|i|)2^{-|i|\gamma/2}. \tag{2.118}$$

Next, fix $N\in\mathbb{N}$ and let $\mathscr{I}$ be an arbitrary finite subset of $\{(l,m)\in\mathbb{Z}\times\mathbb{Z}:\ l,\,m\geq\kappa_E\}$. Then for each function $f\in L^2(E,\sigma)$ with $\|f\|_{L^2(E,\sigma)}=1$, using (2.118) we may estimate

$$\begin{aligned}\Big\|\sum_{(j,k)\in\mathscr{I},\,|j-k|>N}D_jD_kf\Big\|_{L^2(E,\sigma)}&=\Big\|\sum_{i\in\mathbb{Z},\,|i|>N}\Big(\sum_{l\in\mathbb{Z},\,(l+i,l)\in\mathscr{I}}D_{l+i}D_lf\Big)\Big\|_{L^2(E,\sigma)} \qquad (2.119)\\ &\leq\sum_{i\in\mathbb{Z},\,|i|>N}\Big\|\sum_{l\in\mathbb{Z},\,(l+i,l)\in\mathscr{I}}D_{l+i}D_lf\Big\|_{L^2(E,\sigma)}\leq\sum_{i\in\mathbb{Z},\,|i|>N}C(1+|i|)2^{-|i|\gamma/2}\leq C_\gamma N2^{-N\gamma/2},\end{aligned}$$

for some finite constant $C_\gamma>0$ which is independent of N. In turn, based on (2.87), (2.88) and (2.119) we deduce that

$$\begin{gathered}R_N:=\sum_{\substack{|j-k|>N\\ j,k\geq\kappa_E}}D_jD_k\ \text{ converges pointwise unconditionally in } L^2(E,\sigma)\text{, and}\\ \text{there exists } C_\gamma\in(0,\infty)\text{ such that }\ \|R_N\|_{L^2(E,\sigma)\to L^2(E,\sigma)}\leq C_\gamma N2^{-N\gamma/2}.\end{gathered} \tag{2.120}$$

In a similar fashion to (2.118)-(2.120), we may also deduce that

$$T_N:=\sum_{\substack{|j-k|\leq N\\ j,k\geq\kappa_E}}D_jD_k\ \text{ converges pointwise unconditionally in } L^2(E,\sigma). \tag{2.121}$$

Consequently, if we now set

$$D_l^N:=\sum_{\substack{i\in\mathbb{Z},\,|i|\leq N\\ i\geq\kappa_E-l}}D_{l+i},\qquad\text{for each }\ l\in\mathbb{Z}, \tag{2.122}$$

then (cf. (2.76)) the series T_N may be rearranged as

$$T_N=\sum_{l\in\mathbb{Z},\,l\geq\kappa_E}D_lD_l^N, \tag{2.123}$$

where the sum converges pointwise unconditionally in $L^2(E,\sigma)$. Combining (2.115), (2.120) and (2.121), we arrive at the identity

$$\big(I-\mathcal{S}_{\kappa_E}\big)^2=R_N+T_N\quad\text{on}\quad L^2(E,\sigma), \tag{2.124}$$

which is convenient to further re-write as

$$I=R_N+\widetilde{T}_N\quad\text{on}\quad L^2(E,\sigma),\qquad\text{where}\quad\widetilde{T}_N:=T_N+\mathcal{S}_{\kappa_E}\big(2I-\mathcal{S}_{\kappa_E}\big). \tag{2.125}$$

Thanks to the estimate in (2.120), it follows from (2.125) that

$$\widetilde{T}_N : L^2(E,\sigma) \to L^2(E,\sigma) \text{ is boundedly invertible for } N \in \mathbb{N} \text{ sufficiently large.} \tag{2.126}$$

Hence, for N sufficiently large and fixed, based on (2.126) we may write that $I = \widetilde{T}_N(\widetilde{T}_N)^{-1}$, and keeping in mind (2.123) and (2.125), we arrive at the following Calderón-type reproducing formula

$$I = \Big(\sum_{l\in\mathbb{Z},\, l\geq\kappa_E} D_l\widetilde{D}_l\Big) + \mathcal{S}_{\kappa_E}\big(2I - \mathcal{S}_{\kappa_E}\big)(\widetilde{T}_N)^{-1}, \tag{2.127}$$

where the sum converges pointwise unconditionally in $L^2(E,\sigma)$, and

$$\widetilde{D}_l := D_l^N(\widetilde{T}_N)^{-1}, \qquad \forall\, l \in \mathbb{Z} \text{ with } l \geq \kappa_E. \tag{2.128}$$

From this (2.82) follows with $R := \big(\mathcal{S}_{\kappa_E} - 2I\big)(\widetilde{T}_N)^{-1}$. Finally, (2.81) is a consequence of (2.128), the fact that the sum in (2.122) has a finite number of terms, (2.126) and (2.113). $\square$

2.4 Dyadic Carleson tents

Suppose that $(\mathscr{X},\rho)$ is a geometrically doubling quasi-metric space and that E is a nonempty, closed, proper subset of $(\mathscr{X},\tau_\rho)$. It follows from the discussion below Definition 2.5 that $(E,\rho\big|_E)$ is also a geometrically doubling quasi-metric space. We now introduce dyadic Carleson tents in this setting. These are sets in $\mathscr{X}\setminus E$ that are adapted to E in the same way that classical Carleson boxes or tents in the upper-half space $\mathbb{R}^{n+1}_+$ are adapted to $\mathbb{R}^n$. We require a number of preliminaries before we introduce these sets in (2.131) below. First, fix a collection $\mathbb{D}(E)$ of dyadic cubes contained in E as in Proposition 2.12. Second, choose $\lambda \in [2C_\rho,\infty)$ and fix a Whitney covering $\mathbb{W}_\lambda(\mathscr{X}\setminus E)$ of balls contained in $\mathscr{X}\setminus E$ as in Proposition 2.6. Following Convention 2.7, we refer to these $\rho_\#$-balls as Whitney cubes, and for each $I \in \mathbb{W}_\lambda(\mathscr{X}\setminus E)$, we use the notation $\ell(I)$ for the radius of I. Third, choose $C_* \in [1,\infty)$, and for each $Q \in \mathbb{D}(E)$, define the following collection of Whitney cubes:

$$W_Q := \{I \in \mathbb{W}_\lambda(\mathscr{X}\setminus E) :\, C_*^{-1}\ell(I) \leq \ell(Q) \leq C_*\ell(I) \text{ and } \mathrm{dist}_\rho(I,Q) \leq \ell(Q)\}. \tag{2.129}$$

Fourth, for each $Q \in \mathbb{D}(E)$, define the following subset of $(\mathscr{X},\tau_\rho)$:

$$\mathcal{U}_Q := \bigcup_{I\in W_Q} I. \tag{2.130}$$

Since from Theorem 2.2 we know that the regularized quasi-distance $\rho_\#$ is continuous, it follows that the $\rho_\#$-balls are open. As such, that each I in W_Q, hence $\mathcal{U}_Q$ itself, is open.

Finally, for each $Q \in \mathbb{D}(E)$, the `dyadic Carleson tent` $T_E(Q)$ `over` Q is defined as follows:

$$T_E(Q) := \bigcup_{Q'\in\mathbb{D}(E),\, Q'\subseteq Q} \mathcal{U}_{Q'}. \tag{2.131}$$

For most of the subsequent work we will assume that the Whitney covering $\mathbb{W}_\lambda(\mathscr{X}\setminus E)$ and the constant C_* are chosen as in the following lemma.

Lemma 2.21. *Let $(\mathscr{X},\rho)$ be a geometrically doubling quasi-metric space and suppose that E is a nonempty, closed, proper subset of $(\mathscr{X},\tau_\rho)$. Fix a collection $\mathbb{D}(E)$ of dyadic cubes in E as in Proposition 2.12. Next, choose $\lambda \in [2C_\rho,\infty)$, fix a Whitney covering $\mathbb{W}_\lambda(\mathscr{X}\setminus E)$ of $\mathscr{X}\setminus E$, and let Λ denote the constant associated with λ as in Proposition 2.6. Finally, choose*

$$C_* \in [4C_\rho^4\,\Lambda,\infty), \tag{2.132}$$

and define the collection $\{\mathcal{U}_Q\}_{Q\in\mathbb{D}(E)}$ associated with $\mathbb{W}_\lambda(\mathscr{X}\setminus E)$ and C_* as in (2.129)-(2.130).

Then there exists $\epsilon\in(0,1)$, depending only on λ and geometry, with the property that

$$\big\{x\in\mathscr{X}\setminus E:\,\delta_E(x)<\epsilon\,\mathrm{diam}_\rho(E)\big\}\subseteq \bigcup_{Q\in\mathbb{D}(E)}\mathcal{U}_Q. \tag{2.133}$$

Proof. If $\mathrm{diam}_\rho(E)=\infty$, then both sides of (2.133) are equal to $\mathscr{X}\setminus E$ for all $\epsilon\in(0,1)$, since the Whitney cubes cover $\mathscr{X}\setminus E$, so the result is immediate. Now assume that $\mathrm{diam}_\rho(E)<\infty$. Fix some integer $N\in\mathbb{N}$, to be specified later, and consider an arbitrary point $x\in\mathscr{X}\setminus E$ with $\delta_E(x)<2^{-N}\mathrm{diam}_\rho(E)$. Then by (2.8) and (2.48) we have $0<\delta_E(x)<2^{-N-\kappa_E}$, hence there exists $k\in\mathbb{Z}$ with $k\geq\kappa_E$ such that $2^{-N-k-1}\leq\delta_E(x)<2^{-N-k}$. Now, select a ball $I=B_{\rho_\#}(x_I,\ell(I))\in\mathbb{W}_\lambda(\mathscr{X}\setminus E)$ such that $x\in I$. Then, by *(3)* in Proposition 2.6, there exists $z\in E$ such that $\rho_\#(x_I,z)<\Lambda\ell(I)$. Consequently,

$$\delta_E(x)\leq\rho_\#(x,z)\leq C_\rho\,\max\{\rho_\#(x,x_I),\rho_\#(x_I,z)\}<C_\rho\Lambda\ell(I). \tag{2.134}$$

In addition, *(3)* in Proposition 2.6 also gives that $B_{\rho_\#}(x_I,\lambda\ell(I))\subseteq\mathscr{X}\setminus E$ and, hence, for every $y\in E$

$$\begin{aligned}2C_\rho\ell(I) &\leq \lambda\ell(I)\leq\rho_\#(x_I,y)\leq C_\rho\,\rho_\#(x_I,x)+C_\rho\,\rho_\#(x,y)\\ &\leq C_\rho\,\ell(I)+C_\rho\,\rho_\#(x,y).\end{aligned} \tag{2.135}$$

After canceling like-terms in the most extreme sides of (2.135) and taking the infimum over all $y\in E$, we arrive at

$$\ell(I)\leq\delta_E(x). \tag{2.136}$$

Next, since $\delta_E(x)<2^{-N-k}$, there exists $x_0\in E$ such that $\rho_\#(x,x_0)<2^{-N-k}$. Furthermore, by invoking *(7)* in Proposition 2.12 we may choose $Q\in\mathbb{D}_k(E)$ with the property that $B_{\rho_\#}(x_0,2^{-N-k})\cap Q$ contains at least one point x_1. Thus, by (2.14) we have

$$\begin{aligned}\mathrm{dist}_\rho(I,Q) &\leq \mathrm{dist}_\rho(x,Q)\leq\rho(x,x_1)\leq C_\rho^2\rho_\#(x,x_1)\\ &\leq C_\rho^2C_{\rho_\#}\max\{\rho_\#(x,x_0),\rho_\#(x_0,x_1)\}<C_\rho^3 2^{-N-k}=C_\rho^3 2^{-N}\ell(Q).\end{aligned} \tag{2.137}$$

This shows that

$$2^N>C_\rho^3\Longrightarrow\mathrm{dist}_\rho(I,Q)\leq\ell(Q). \tag{2.138}$$

Starting with (2.136) and keeping in mind that $\delta_E(x)<2^{-N-k}$, we obtain

$$\ell(I)<2^{-N-k}=2^{-N}\ell(Q)\leq\ell(Q). \tag{2.139}$$

Finally, with the help of (2.134) we write $2^{-N-1}\ell(Q) = 2^{-N-k-1} \leq \delta_E(x) \leq C_\rho \Lambda \ell(I)$ which further entails

$$C_* \geq 2^{N+1} C_\rho \Lambda \Longrightarrow \ell(I) \geq C_*^{-1} \ell(Q). \tag{2.140}$$

At this stage, by choosing $N \in \mathbb{N}$ such that

$$N - 1 \leq \log_2\big(C_\rho^3\big) < N, \tag{2.141}$$

we may conclude from (2.138), (2.139) and (2.140) that $I \in W_Q$ when $C_* \geq 4C_\rho^4 \Lambda$. This, in turn, forces $x \in I \subseteq \mathcal{U}_Q$. Taking $\epsilon := 2^{-N}$ with N as in (2.141) then justifies (2.133), and finishes the proof of the lemma. □

We now return to the context introduced in the first paragraph of this subsection, and in particular, where $\lambda \in [2C_\rho, \infty)$ and $C_* \in [1, \infty)$. For further reference, we note that then there exists $C_o \in [1, \infty)$ such that

$$C_o^{-1} \ell(Q) \leq \delta_E(x) \leq C_o \ell(Q), \qquad \forall\, Q \in \mathbb{D}(E) \text{ and } \forall\, x \in \mathcal{U}_Q. \tag{2.142}$$

Indeed, an inspection of (2.134), (2.136), (2.129) and (2.130) shows that (2.142) holds when

$$C_o := C_* C_\rho \Lambda, \tag{2.143}$$

where Λ is the constant associated with λ as in Proposition 2.6.

The reader should be aware of the fact that even when (2.133) holds it may happen that some $\mathcal{U}_Q$'s are empty. However, under the assumption that $(\mathscr{X}, \rho, \mu)$ is an m-dimensional ADR space and granted the existence of a measure σ such that $(E, \rho|_E, \sigma)$ becomes a d-dimensional ADR space for some $d \in (0, m)$, matters may be arranged so that this eventuality never materializes. In particular, if C_* is large enough (depending on λ and geometry), then $\mathcal{U}_Q \neq \emptyset$ for all $Q \in \mathbb{D}(E)$. This is a simple consequence of the following lemma, which is proved in [61].

Lemma 2.22. *Let $(\mathscr{X}, \rho, \mu)$ be an m-dimensional* ADR *space, for some $m > 0$, and assume that E is a closed subset of $(\mathscr{X}, \tau_\rho)$ with the property that there exists a measure σ on E for which $(E, \rho|_E, \sigma)$ is a d-dimensional* ADR *space for some $d \in (0, m)$.*

Then there exists $\vartheta \in (0, 1)$ such that for each $x_0 \in \mathscr{X}$ and each finite $r \in (0, \mathrm{diam}_\rho(\mathscr{X})]$ one may find $x \in \mathscr{X}$ with the property that $B_\rho(x, \vartheta r) \subseteq B_\rho(x_0, r) \setminus E$.

We again return to the context introduced in the first paragraph of this subsection, and in particular, where $\lambda \in [2C_\rho, \infty)$ and $C_* \in [1, \infty)$. For each $Q \in \mathbb{D}(E)$, recall the dyadic Carleson tent $T_E(Q)$ over Q from (2.131):

$$T_E(Q) := \bigcup_{Q' \in \mathbb{D}(E),\ Q' \subseteq Q} \mathcal{U}_{Q'}. \tag{2.144}$$

A property that will be needed later is the fact that

$$\begin{gathered} \text{there exists } C \in (0, \infty) \text{ depending only on } C_* \text{ from (2.129) and } \rho \\ \text{such that } T_E(Q) \subseteq B_\rho\big(x, C\ell(Q)\big) \setminus E, \quad \forall\, Q \in \mathbb{D}(E),\ \forall\, x \in Q. \end{gathered} \tag{2.145}$$

Indeed, if $Q \in \mathbb{D}(E)$ and $y \in T_E(Q)$ are arbitrary, then there exists $Q' \in \mathbb{D}(E)$, $Q' \subseteq Q$ such that $y \in I$, for some $I \in W_{Q'}$. Hence, for each $x \in Q$, we have

$$\begin{aligned} \rho(y,x) &\leq C\mathrm{diam}_\rho(I) + C\mathrm{dist}_\rho(I,Q') + C\mathrm{diam}_\rho(Q) \\ &\leq C\ell(Q') + C\ell(Q) \leq C\ell(Q), \end{aligned} \tag{2.146}$$

where C is a finite positive geometric constant. Now (2.145) follows from (2.146).

The following lemma compliments the containment in (2.145).

Lemma 2.23. *Assume all of the hypotheses contained in the first paragraph of Lemma 2.21 and recall the family of dyadic Carleson tents* $\{T_E(Q)\}_{Q\in\mathbb{D}(E)}$ *defined in* (2.131).

Then there exists $\varepsilon \in (0,1)$*, depending only on* λ *and geometry, with the property that*

$$B_{\rho_\#}\big(x_Q, \varepsilon\ell(Q)\big) \setminus E \subseteq T_E(Q), \qquad \forall\, Q \in \mathbb{D}(E). \tag{2.147}$$

Proof. Fix $\varepsilon \in (0,1)$ to be specified later and let N be as in (2.141). Also take an arbitrary $Q \in \mathbb{D}(E)$ and fix $x \in B_{\rho_\#}\big(x_Q, \varepsilon\ell(Q)\big) \setminus E$. Then $\rho_\#(x_Q, x) < \varepsilon\ell(Q)$ and making the restriction

$$\varepsilon < 2^{-N-1} \tag{2.148}$$

we have

$$\delta_E(x) \leq \rho_\#(x_Q, x) < \varepsilon\ell(Q) \leq 2\varepsilon\,\mathrm{diam}_\rho(E) < 2^{-N}\mathrm{diam}_\rho(E). \tag{2.149}$$

Thus, all considerations in the first part of the proof of Lemma 2.21 up to (2.136) apply. In particular, it follows that $\delta_E(x) < \min\{2^{-N-k}, \varepsilon\ell(Q)\}$. Hence, there exists $x_0 \in E$ such that $\rho_\#(x, x_0) < \min\{2^{-N-k}, \varepsilon\ell(Q)\}$. Applying property *(7)* in Proposition 2.12, we may choose $Q' \in \mathbb{D}_k(E)$ such that $B_{\rho_\#}\big(x_0, \varepsilon\ell(Q)\big) \cap Q' \neq \emptyset$. At this point we make the claim that

$$B_{\rho_\#}\big(x_0, \varepsilon\ell(Q)\big) \cap E \subseteq Q \quad \text{ if } \varepsilon \text{ is sufficiently small.} \tag{2.150}$$

Indeed, first observe that

$$\rho_\#(x_0, x_Q) \leq C_\rho \max\{\rho_\#(x_0, x), \rho_\#(x, x_Q)\} < \varepsilon C_\rho \ell(Q). \tag{2.151}$$

Consequently, if $y \in B_{\rho_\#}\big(x_0, \varepsilon\ell(Q)\big) \cap E$ is arbitrary, then

$$\rho_\#(x_Q, y) \leq C_\rho \max\{\rho_\#(x_Q, x_0), \rho_\#(x_0, y)\} < \varepsilon C_\rho^2 \ell(Q). \tag{2.152}$$

Property (2.50) ensures that $B_{\rho_\#}\big(x_Q, a_0 C_\rho^{-2}\ell(Q)\big) \cap E \subseteq Q$, so (2.152) implies that $y \in Q$ if

$$\varepsilon < a_0 C_\rho^{-4}, \tag{2.153}$$

proving the claim in (2.150). In turn, if we assume that ε is sufficiently small, the inclusion in (2.150) implies

$$Q' \cap Q \neq \emptyset. \tag{2.154}$$

On the other hand, using the reasoning in the proof of Lemma 2.21 that yielded (2.137)-(2.141), this time with Q' replacing Q, we obtain that if N is as in (2.141) (recall that we are assuming that C_* satisfies (2.132)), then

$$x \in I \subseteq \mathcal{U}_{Q'}. \tag{2.155}$$

Thus, using also (2.142), we have

$$\ell(Q') \le C_o \delta_E(x) \le C_o \rho_\#(x_Q, x) < C_o \varepsilon \ell(Q). \tag{2.156}$$

Hence, under the additional restriction $\varepsilon < C_o^{-1}$, we arrive at the conclusion that $Q' \in \mathbb{D}_j(E)$ for some $j > k$, which when combined with (2.154) and *(3)* in Proposition 2.12, forces $Q' \subseteq Q$. This in concert with (2.155) and (2.131), shows that $x \in T_E(Q)$ provided

$$0 < \varepsilon < \min\{2^{-N-1}, a_0 C_\rho^{-4}, C_o^{-1}\}. \tag{2.157}$$

The proof of the lemma is now complete. □

Next we prove a finite overlap property for the sets in $\{\mathcal{U}_Q\}_{Q \in \mathbb{D}(E)}$ from (2.130). Throughout the manuscript, we agree that $\mathbf{1}_A$ stands for the characteristic (or indicator) function of the set A.

Lemma 2.24. *Let $(\mathscr{X}, \rho)$ be a geometrically doubling quasi-metric space and suppose that E is a nonempty, closed, proper subset of $(\mathscr{X}, \tau_\rho)$. Fix $a \in [1, \infty)$, a collection $\mathbb{D}(E)$ of dyadic cubes in E as in Proposition 2.12, and $C_* \in [1, \infty)$.*

If $\lambda \in [a, \infty)$, and we fix a Whitney covering $\mathbb{W}_\lambda(\mathscr{X} \setminus E)$ of $\mathscr{X} \setminus E$ as in Proposition 2.6, then there exists $N \in \mathbb{N}$, depending only on λ, C_* and geometry, such that

$$\sum_{Q \in \mathbb{D}(E)} \mathbf{1}_{\mathcal{U}_Q^*} \le N, \tag{2.158}$$

where $\{\mathcal{U}_Q\}_{Q \in \mathbb{D}(E)}$ is the collection associated with $\mathbb{W}_\lambda(\mathscr{X} \setminus E)$ and C_* as in (2.129)-(2.130), *and for each $Q \in \mathbb{D}(E)$, the set (compare with* (2.130)*)*

$$\mathcal{U}_Q^* := \bigcup_{I \in W_Q} aI. \tag{2.159}$$

Proof. Let $\mathbb{D}(E)$ be the collection of dyadic cubes obtained by applying Proposition 2.12. Fix $a \in [1, \infty)$ and consider a Whitney covering $\mathbb{W}_\lambda(\mathscr{X} \setminus E)$ as in Proposition 2.6 with $\lambda \in [a, \infty)$. In particular,

$$\sum_{I \in \mathbb{W}_\lambda(\mathscr{X} \setminus E)} \mathbf{1}_{\lambda I} \le N_1, \qquad \text{for some } N_1 \in \mathbb{N}. \tag{2.160}$$

To proceed, define

$$\mathcal{I} := \bigcup_{Q \in \mathbb{D}(E)} W_Q \subseteq \mathbb{W}_\lambda(\mathscr{X} \setminus E) \tag{2.161}$$

and, for each $I \in \mathcal{I}$,

$$q_I := \{Q \in \mathbb{D}(E) : I \in W_Q\}. \tag{2.162}$$

Then, using (2.160), we estimate

$$\begin{aligned} \sum_{Q \in \mathbb{D}(E)} \mathbf{1}_{\mathcal{U}_Q^*} &\le \sum_{Q \in \mathbb{D}(E)} \sum_{I \in W_Q} \mathbf{1}_{\lambda I} = \sum_{I \in \mathcal{I}} (\# q_I) \cdot \mathbf{1}_{\lambda I} \\ &\le \Big(\sup_{I \in \mathbb{W}_\lambda(\mathscr{X} \setminus E)} \# q_I \Big) \sum_{I \in \mathbb{W}_\lambda(\mathscr{X} \setminus E)} \mathbf{1}_{\lambda I} \le N_1 \cdot \Big(\sup_{I \in \mathbb{W}_\lambda(\mathscr{X} \setminus E)} \# q_I \Big). \end{aligned} \tag{2.163}$$

Hence, once we show that there exists $N_2 \in \mathbb{N}$ such that

$$\# q_I \leq N_2, \qquad \forall\, I \in \mathbb{W}_\lambda(\mathscr{X} \setminus E), \tag{2.164}$$

the desired estimate, (2.158), follows with $N := N_1 N_2$. To prove (2.164), fix an arbitrary $I \in \mathbb{W}_\lambda(\mathscr{X} \setminus E)$ and assume that $Q \in \mathbb{D}(E)$ is such that $I \in W_Q$. Then, from (2.129) we deduce that

$$C_*^{-1}\ell(I) \leq \ell(Q) \leq C_*\ell(I) \quad \text{and} \quad \mathrm{dist}_\rho(I, Q) \leq C_*\ell(I). \tag{2.165}$$

Now, (2.164) follows from (2.165) and the fact that $\big(E, \rho\big|_E\big)$ is geometrically doubling. $\square$

3 $T(1)$ and local $T(b)$ Theorems for Square Functions

This section consists of two parts, dealing with a $T(1)$ Theorem and a local $T(b)$ Theorem for square functions on sets of arbitrary co-dimension, relative to an ambient quasi-metric space (the notion of dimension refers to the degree of Ahlfors-David regularity). The $T(1)$ Theorem generalizes the Euclidean co-dimension one result proved by M. Christ and J.-L. Journé in [15] (cf. also [13, Theorem 20, p. 69]). The local $T(b)$ Theorem generalizes the Euclidean co-dimension one result that was implicit in the solution of the Kato problem in [45, 41, 3], and formulated explicitly in [2, 39, 46].

We consider the following context. Fix two real numbers d, m such that $0 < d < m$, an m-dimensional ADR space $(\mathscr{X}, \rho, \mu)$, a closed subset E of $(\mathscr{X}, \tau_\rho)$, and a Borel measure σ on $(E, \tau_{\rho|_E})$ with the property that $(E, \rho\big|_E, \sigma)$ is a d-dimensional ADR space. In this context, suppose that

$$\begin{gathered} \theta : (\mathscr{X} \setminus E) \times E \longrightarrow \mathbb{R} \quad \text{is Borel measurable with respect to} \\ \text{the relative topology induced by the product topology } \tau_\rho \times \tau_\rho \text{ on } (\mathscr{X} \setminus E) \times E, \end{gathered} \tag{3.1}$$

and has the property that there exist finite positive constants C_θ, α, υ, and $a \in [0, \upsilon)$ such that for all $x \in \mathscr{X} \setminus E$ and $y \in E$ the following hold:

$$|\theta(x, y)| \leq \frac{C_\theta}{\rho(x, y)^{d+\upsilon}} \Big(\frac{\mathrm{dist}_\rho(x, E)}{\rho(x, y)}\Big)^{-a}, \tag{3.2}$$

$$\begin{gathered} |\theta(x, y) - \theta(x, \widetilde{y})| \leq C_\theta \frac{\rho(y, \widetilde{y})^\alpha}{\rho(x, y)^{d+\upsilon+\alpha}} \Big(\frac{\mathrm{dist}_\rho(x, E)}{\rho(x, y)}\Big)^{-a-\alpha}, \\ \forall\, \widetilde{y} \in E \ \text{ with } \ \rho(y, \widetilde{y}) \leq \tfrac{1}{2}\rho(x, y). \end{gathered} \tag{3.3}$$

Then define the integral operator Θ for all functions $f \in L^p(E, \sigma)$, $1 \leq p \leq \infty$, by

$$(\Theta f)(x) := \int_E \theta(x, y) f(y)\, d\sigma(y), \qquad \forall\, x \in \mathscr{X} \setminus E. \tag{3.4}$$

It follows from Hölder's inequality and Lemma 3.5 that the integral in (3.4) is absolutely convergent for each $x \in \mathscr{X} \setminus E$.

Remark 3.1. *The factors in parentheses in* (3.2)-(3.3) *are greater than or equal to* 1*, since for every $x \in \mathscr{X} \setminus E$ and every $y \in E$ we have $\rho(x, y) \geq \mathrm{dist}_\rho(x, E) > 0$, hence* (3.2)-(3.3) *are less demanding than their respective versions in which these factors are omitted.*

We proceed to prove square function versions of the $T(1)$ Theorem and the local $T(b)$ Theorem for the integral operator Θ. As usual, we prove the local $T(b)$ Theorem by verifying the hypotheses of the $T(1)$ Theorem, to which we now turn.

3.1 An arbitrary codimension $T(1)$ theorem for square functions

The main result in this subsection is a $T(1)$ theorem for square functions, to the effect that *a square function estimate for the integral operator Θ holds if and only if $|\Theta(1)|^2$, appropriately weighted by a power of the distance to E, is the density (relative to μ) of a Carleson measure on $\mathscr{X}\setminus E$.* To state this formally, the reader is advised to recall the dyadic cube grid from Proposition 2.12 and the regularized distance function to a set from (2.19).

Theorem 3.2. *Let d, m be two real numbers such that $0 < d < m$. Assume that $(\mathscr{X}, \rho, \mu)$ is an m-dimensional* ADR *space, E is a closed subset of $(\mathscr{X}, \tau_\rho)$, and σ is a Borel regular measure on $(E, \tau_{\rho|_E})$ with the property that $(E, \rho\big|_E, \sigma)$ is a d-dimensional* ADR *space.*

Suppose that Θ is the integral operator defined in (3.4) *with a kernel θ as in* (3.1), (3.2), (3.3). *Furthermore, let $\mathbb{D}(E)$ denote a dyadic cube structure on E, consider a Whitney covering $\mathbb{W}_\lambda(\mathscr{X}\setminus E)$ of $\mathscr{X}\setminus E$ and a constant C_* as in Lemma 2.21 and, corresponding to these, recall the dyadic Carleson tents from* (2.131).

In this context, if

$$\sup_{Q\in\mathbb{D}(E)}\left(\tfrac{1}{\sigma(Q)}\int_{T_E(Q)}|\Theta 1(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\right)<\infty, \tag{3.5}$$

then there exists a finite constant $C>0$ depending only on the constants C_θ, the ADR *constants of E and $\mathscr{X}$, and the value of the supremum in* (3.5), *such that for each function $f\in L^2(E,\sigma)$ one has*

$$\int_{\mathscr{X}\setminus E}|(\Theta f)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\le C\int_E|f(x)|^2\,d\sigma(x),\qquad\forall\, f\in L^2(E,\sigma). \tag{3.6}$$

Finally, the converse of the implication discussed above is also true. In fact, the following stronger claim holds: under the original background assumptions, except that the regularity requirement (3.3) *is now dropped, the fact that*

$$\int_{\substack{x\in\mathscr{X}\\ 0<\delta_E(x)<\eta\,\mathrm{diam}_\rho(E)}}|(\Theta f)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\le C\int_E|f(x)|^2\,d\sigma(x),\quad\forall\, f\in L^2(E,\sigma), \tag{3.7}$$

holds for some $\eta\in(0,\infty)$ implies that (3.5) *holds as well.*

Before presenting the actual proof of Theorem 3.2 we shall discuss a number of preliminary lemmas, starting with the following discrete Carleson estimate.

Lemma 3.3. *Assume (E,ρ,σ) is a space of homogeneous type with the property that σ is Borel regular, and denote by $\mathbb{D}(E)$ a dyadic cube structure on E. If a sequence $\{B_Q\}_{Q\in\mathbb{D}(E)}\subseteq[0,\infty]$ satisfies the discrete Carleson condition*

$$C:=\sup_{R\in\mathbb{D}(E)}\Big[\frac{1}{\sigma(R)}\sum_{Q\in\mathbb{D}(E),\,Q\subseteq R}B_Q\Big]<\infty, \tag{3.8}$$

then for every sequence $\{A_Q\}_{Q\in\mathbb{D}(E)}\subseteq\mathbb{R}$ one has

$$\sum_{Q\in\mathbb{D}(E)}A_QB_Q\le C\int_E A^*\,d\sigma, \tag{3.9}$$

where $A^* : E \to [0,\infty]$ is the function defined by

$$A^*(x) := 0 \quad \text{if} \quad x \in E \setminus \bigcup_{Q\in\mathbb{D}(E)} Q \qquad \text{and} \qquad A^*(x) := \sup_{Q\in\mathbb{D}(E),\, x\in Q} |A_Q| \quad \text{if} \quad x \in \bigcup_{Q\in\mathbb{D}(E)} Q. \tag{3.10}$$

Proof. For each $t > 0$ define $\mathcal{O}_t := \{x \in E : A^*(x) > t\}$. Then it is immediate from definitions that $\mathcal{O}_t = \bigcup_{Q\in\mathbb{D}(E),\, A_Q>t} Q$ for every $t > 0$. This shows that $\mathcal{O}_t$ is open in (E, τ_ρ) (cf. *(1)* in Proposition 2.12) and, hence, A^* is σ-measurable. Note that if $A^* \in L^1(E,\sigma)$ (otherwise there is nothing to prove), then by Tschebyshev's inequality,

$$\sigma(\mathcal{O}_t) \leq \frac{1}{t}\int_E A^*(x)\, d\sigma(x) < \infty, \qquad \forall\, t > 0. \tag{3.11}$$

This ensures that for each $t > 0$ we may meaningfully define $D_t \subseteq \mathbb{D}(E)$, the collection of maximal dyadic cubes contained in $\mathcal{O}_t$, i.e.,

$$D_t := \big\{R \in \mathbb{D}(E) : \ R \subseteq \mathcal{O}_t \text{ and } \nexists\, Q \in \mathbb{D}(E) \text{ such that } R \subseteq Q \subseteq \mathcal{O}_t \text{ and } R \neq Q\big\}. \tag{3.12}$$

The cubes in D_t are pairwise disjoint, and

$$\mathcal{O}_t = \bigcup_{R\in D_t} R. \tag{3.13}$$

Now for each $Q \in \mathbb{D}(E)$ define

$$h_Q : (0,\infty) \longrightarrow \mathbb{R}, \qquad h_Q(t) := \begin{cases} 1, & \text{if } 0 < t < A_Q, \\ 0, & \text{otherwise.} \end{cases} \tag{3.14}$$

Then, for each $t > 0$ we have

$$\begin{aligned} \sum_{Q\in\mathbb{D}(E)} h_Q(t) B_Q &= \sum_{Q\in\mathbb{D}(E),\, Q\subseteq\mathcal{O}_t} B_Q = \sum_{R\in D_t}\Bigg(\sum_{Q\in\mathbb{D}(E),\, Q\subseteq R} B_Q\Bigg) \\ &\leq C\sum_{R\in D_t}\sigma(R) = C\sigma(\mathcal{O}_t), \end{aligned} \tag{3.15}$$

where for the first inequality in (3.15) we have used (3.8), while the last equality follows from (3.13). Hence,

$$\begin{aligned} \sum_{Q\in\mathbb{D}(E)} A_Q B_Q &= \int_0^\infty \sum_{Q\in\mathbb{D}(E)} h_Q(t) B_Q\, dt \leq C\int_0^\infty \sigma(\mathcal{O}_t)\, dt \\ &= C\int_0^\infty\int_E \mathbf{1}_{\{A^*>t\}}(x)\, d\sigma(x)\, dt = C\int_E\int_0^\infty \mathbf{1}_{\{A^*>t\}}(x)\, dt\, d\sigma(x) \\ &= C\int_E A^*(x)\, d\sigma(x), \end{aligned} \tag{3.16}$$

completing the proof of the lemma. $\square$

We continue by recording a quantitative version of the classical Urysohn lemma in the context of Hölder functions on quasi-metric spaces from [60] (cf. also [1] for a refinement).

Lemma 3.4. *Let (E,ρ) be a quasi-metric space and assume that β is a real number with the property that $0<\beta\leq[\log_2 C_\rho]^{-1}$. Assume that $F_0,F_1\subseteq E$ are two nonempty sets with the property that $\mathrm{dist}_\rho(F_0,F_1)>0$. Then, there exists a function $\eta:E\to\mathbb{R}$ such that*

$$0\leq\eta\leq 1 \quad on \quad E, \quad \eta\equiv 0 \quad on \quad F_0, \quad \eta\equiv 1 \quad on \quad F_1, \tag{3.17}$$

and for which there exists a finite constant $C>0$, depending only on ρ, such that

$$\sup_{\substack{x,y\in E\\ x\neq y}}\frac{|\eta(x)-\eta(y)|}{\rho(x,y)^\beta}\leq C\big(\mathrm{dist}_\rho(F_0,F_1)\big)^{-\beta}. \tag{3.18}$$

In the proof of Theorem 3.2 we shall also need a couple of results of geometric measure theoretic nature, which we next discuss.

Lemma 3.5. *Let $(\mathscr{X},\rho)$ be a quasi-metric space. Suppose $E\subseteq\mathscr{X}$ is nonempty and σ is a measure on E such that $(E,\rho\big|_E,\sigma)$ becomes a d-dimensional* ADR *space, for some $d>0$. Fix a real number $m>d$. Then there exists $C\in(0,\infty)$ depending only on m, ρ, and the* ADR *constant of E such that*

$$\int_E\frac{1}{\rho_\#(x,y)^m}\,d\sigma(y)\leq C\delta_E(x)^{d-m}, \qquad \forall\,x\in\mathscr{X}\setminus E. \tag{3.19}$$

Also, for each $\varepsilon>0$ and $c>0$, there exists $C\in(0,\infty)$ depending only on ε, c, ρ, and the ADR *constant of E such that for every σ-measurable function $f:E\to[0,\infty]$ one has*

$$\int_{y\in E,\,\rho_\#(y,x)>cr}\frac{r^\varepsilon}{\rho_\#(y,x)^{d+\varepsilon}}f(y)\,d\sigma(y)\leq C\,M_E(f)(x) \qquad \forall\,x\in E, \quad \forall\,r>0, \tag{3.20}$$

where M_E is as in (2.103)*.*

Proof. Fix $x\in\mathscr{X}\setminus E$. Then

$$\begin{aligned}\int_E\frac{1}{\rho_\#(y,x)^m}\,d\sigma(y) &\leq \int_E\mathbf{1}_{\{z:\,\rho_\#(z,x)\geq\delta_E(x)\}}(y)\frac{1}{\rho_\#(y,x)^m}\,d\sigma(y)\\ &= C\sum_{j=0}^\infty\int_E\mathbf{1}_{\{z:\,\rho_\#(z,x)\in[2^j\delta_E(x),2^{j+1}\delta_E(x))\}}(y)\frac{1}{\rho_\#(y,x)^m}\,d\sigma(y)\\ &\leq C\sum_{j=0}^\infty\frac{1}{[2^j\delta_E(x)]^m}\sigma\left(B_{\rho_\#}(x,2^{j+1}\delta_E(x))\cap E\right)\\ &\leq C\sum_{j=0}^\infty\frac{1}{[2^j\delta_E(x)]^m}\left[2^{j+1}\delta_E(x)\right]^d\\ &= C\delta_E(x)^{d-m},\end{aligned} \tag{3.21}$$

where for the last inequality in (3.21) we have used the fact that $(E,\rho_\#|_E,\sigma)$ is a d-dimensional ADR space, while the last equality uses the condition $m>d$. This concludes the proof of (3.19).

Finally, (3.20) is proved similarly, by decomposing the domain of integration in dyadic annuli centered at x, at scale r, and then using the fact that $(E, \rho_{\#}|_E, \sigma)$ is a d-dimensional ADR space. $\square$

For a proof of our second result of geometric measure theoretic nature the interested reader is referred to [61], where more general results of this type are established.

Lemma 3.6. *Assume that $(\mathscr{X}, \rho, \mu)$ is an m-dimensional* ADR *space for some $m > 0$ and let $E \subseteq \mathscr{X}$ be nonempty, closed. Suppose there exists a measure σ on E such that $(E, \rho\big|_E, \sigma)$ is a d-dimensional* ADR *space with $0 < d < m$. If $\gamma < m - d$, then there exists a finite positive constant C_0 which depends only on γ and the* ADR *constants of E and $\mathscr{X}$, such that*

$$\int\limits_{x \in B_\rho(x_0, R),\, \delta_E(x) < r} \delta_E(x)^{-\gamma}\, d\mu(x) \leq C_0\, r^{m-d-\gamma} R^d, \tag{3.22}$$

for every $x_0 \in E$ and every $r, R > 0$.

At this stage, we are ready to present the

Proof of Theorem 3.2. For notational simplicity, abbreviate $A_Q f := ⨍_Q f\, d\sigma$, for every cube $Q \in \mathbb{D}(E)$ whenever $f : E \to \mathbb{C}$ is locally integrable. Also, recall our convention that $\delta_E(x)$ stands for $\mathrm{dist}_{\rho_\#}(x, E)$, for every $x \in \mathscr{X} \setminus E$, and recall the Hardy-Littlewood maximal operator M_E from (2.103). We break down the proof of the implication "(3.5)$\Rightarrow$(3.6)" into a number of steps.

`Step I.` *We claim that for every $r \in (1, \infty)$ there exist finite positive constants C and β such that for each $l, k \in \mathbb{Z}$ with $l, k \geq \kappa_E$ and every $Q \in \mathbb{D}_k(E)$, fixed thereafter, the following inequality holds*:

$$\sup_{x \in \mathcal{U}_Q} \Big| \delta_E(x)^{\upsilon} \big(\Theta(D_l g)(x) - (\Theta 1)(x) A_Q(D_l g) \big) \Big| \leq C 2^{-|k-l|\beta} \inf_{w \in Q} \Big[M_E^2(|g|^r)(w) \Big]^{\frac{1}{r}}, \tag{3.23}$$

for every $g : E \to \mathbb{R}$ locally integrable. Here, D_l is the operator introduced in (2.80), whose integral kernel is denoted by $h_l(\cdot, \cdot)$ (cf. the discussion in the proof of Proposition 2.18).

To justify (3.23), fix $k \in \mathbb{Z}$ with $k \geq \kappa_E$, $Q \in \mathbb{D}_k(E)$, and pick a number $k_0 \in \mathbb{N}_0$ to be specified later, purely in terms of geometrical constants. We distinguish two cases.

Case I: $k + k_0 \geq l$. As a preamble, we remark that $k + k_0 \geq l$ forces

$$2^{-(k+k_0-l)} \approx 2^{-|k-l|}, \tag{3.24}$$

where the comparability constants depend only on k_0. Indeed, observe that if $j \in \mathbb{R}$ is such that $j \geq -k_0$, then $j \leq |j| \leq j + 2k_0$, hence $2^{-j} \approx 2^{-|j|}$. Taking now $j := k - l$, (3.24) follows.

Turning now to the proof of (3.23) in earnest, using Fubini's Theorem, for each $l \in \mathbb{Z}$, we

write

$$\begin{aligned}
&\delta_E(x)^\upsilon\big(\Theta(D_l g)(x)-(\Theta 1)(x)A_Q(D_l g)\big)\\
&\qquad=\delta_E(x)^\upsilon\int_E\theta(x,y)\int_E h_l(y,z)g(z)\,d\sigma(z)\,d\sigma(y)\\
&\qquad\quad-\delta_E(x)^\upsilon(\Theta 1)(x)\fint_Q\int_E h_l(y,z)g(z)\,d\sigma(z)\,d\sigma(y)\\
&\qquad=\int_E\Big[\int_E\Phi(x,y)h_l(y,z)\,d\sigma(y)\Big]\,g(z)\,d\sigma(z),\qquad\forall\,x\in\mathcal{U}_Q,
\end{aligned}\tag{3.25}$$

where

$$\Phi(x,y):=\delta_E(x)^\upsilon\Big[\theta(x,y)-\tfrac{1}{\sigma(Q)}(\Theta 1)(x)\mathbf{1}_Q(y)\Big],\quad\forall\,x\in\mathscr{X}\setminus E,\quad\forall\,y\in E.\tag{3.26}$$

Note that, by design,

$$\int_E\Phi(x,y)\,d\sigma(y)=0,\qquad\forall\,x\in\mathscr{X}\setminus E,\tag{3.27}$$

and we claim that

$$|\Phi(x,y)|\leq\frac{C}{\sigma(Q)},\qquad\forall\,x\in\mathcal{U}_Q,\quad\forall\,y\in E.\tag{3.28}$$

Indeed, if $x\in\mathcal{U}_Q$ then $\delta_E(x)\approx\ell(Q)$ (with constants independent of x) and making use of (3.2) and the fact that $\delta_E(\cdot)\approx\operatorname{dist}_\rho(\cdot,E)$ (see *(4)* in Theorem 2.2), we obtain

$$\delta_E(x)^\upsilon|\theta(x,y)|\leq\frac{C\delta_E(x)^{\upsilon-a}}{\rho(x,y)^{d+\upsilon-a}}\leq\frac{C\delta_E(x)^{\upsilon-a}}{\delta_E(x)^{d+\upsilon-a}}\leq\frac{C}{\ell(Q)^d}\leq\frac{C}{\sigma(Q)},\qquad\forall\,y\in E.\tag{3.29}$$

In addition,

$$\delta_E(x)^\upsilon|(\Theta 1)(x)|\leq C\delta_E(x)^{\upsilon-a}\int_{y\in E}\frac{d\sigma(y)}{\rho_\#(x,y)^{d+\upsilon-a}}\leq C,\qquad\forall\,x\in\mathcal{U}_Q,\tag{3.30}$$

where for the last inequality in (3.30) we made use of (3.19). Now (3.28) follows from (3.26), (3.29) and (3.30).

Denote by x_Q the center of Q and let $\varepsilon\in(0,1)$ and $C_0>0$ be fixed, to be specified later. Then for every $w\in Q$ fixed, due to (3.27) for each $z\in E$ we may write

$$\begin{aligned}
&\Big|\int_E\Phi(x,y)h_l(y,z)\,d\sigma(y)\Big|=\Big|\int_E\Phi(x,y)[h_l(y,z)-h_l(w,z)]\,d\sigma(y)\Big|\\
&\qquad\leq\int\limits_{y\in E,\,\rho_\#(y,x_Q)\leq C_0 2^{(k+k_0-l)\varepsilon}\ell(Q)}|\Phi(x,y)||h_l(y,z)-h_l(w,z)|\,d\sigma(y)\\
&\qquad\quad+\int\limits_{y\in E,\,\rho_\#(y,x_Q)>C_0 2^{(k+k_0-l)\varepsilon}\ell(Q)}|\Phi(x,y)||h_l(y,z)-h_l(w,z)|\,d\sigma(y)\\
&\qquad=:I_1+I_2.
\end{aligned}\tag{3.31}$$

In order to estimate I_1 we make the claim that if $\widetilde{C}$ is chosen large enough (compared to finite positive background constants and C_0) then

$$
\begin{aligned}
&|h_l(y,z)-h_l(w,z)| \le C\,2^{l(d+\gamma)}2^{(k+k_0-l)\varepsilon\gamma}\ell(Q)^\gamma \mathbf{1}_{\{\rho_\#(w,\cdot)\le\widetilde{C}2^{-l}\}}(z)\\
&\text{whenever } \ z\in E,\ y\in E,\ w\in Q \ \text{ and } \ \rho_\#(y,x_Q)\le C_0 2^{(k+k_0-l)\varepsilon}\ell(Q),
\end{aligned}
\tag{3.32}
$$

with γ as in (2.104). To justify this claim, first note that if C_0 is large, then since $k+k_0-l\ge 0$, we have

$$
y\in E,\ w\in Q \text{ and } \rho_\#(y,x_Q)\le C_0 2^{(k+k_0-l)\varepsilon}\ell(Q) \Rightarrow \rho_\#(y,w)\le CC_0 2^{(k+k_0-l)\varepsilon}\ell(Q). \tag{3.33}
$$

From now on, assume that C_0 is large enough to ensure the validity of (3.33). Second, if y,w are as in (3.33) and $z\in E$ is such that $\rho_\#(z,w)\ge\widetilde{C}2^{-l}$, then

$$
\begin{aligned}
\widetilde{C}2^{-l} &\le \rho_\#(z,w)\le C(\rho_\#(z,y)+\rho_\#(y,w))\le C\rho_\#(z,y)+CC_0 2^{(k+k_0-l)\varepsilon}\ell(Q)\\
&\le C\rho_\#(z,y)+2^{k_0}C_0C2^{-l},
\end{aligned}
\tag{3.34}
$$

for some finite geometric constant $C>0$. Now choosing $\widetilde{C}:=2^{k_0+1}C_0C$ (which is permissible since, in the end, the parameter $k_0\in\mathbb{N}_0$ is chosen to depend only on finite positive background geometrical constants) we may absorb $2^{k_0}C_0C2^{-l}$ into $\widetilde{C}2^{-l}$ yielding (with $C_1:=2^{k_0}C_0C$)

$$
\left.\begin{aligned}
y\in E,\ w\in Q,\ \rho_\#(y,x_Q)\le C_0 2^{(k+k_0-l)\varepsilon}\ell(Q)\\
\text{and } \ \rho_\#(z,w)\ge\widetilde{C}2^{-l}
\end{aligned}\right\}\Longrightarrow \rho_\#(z,y)>C_1 2^{-l}. \tag{3.35}
$$

Moreover, we can further increase C_0 and, in turn, $\widetilde{C}$ to insure that the constant C_1 in the last inequality in (3.35) is larger that the constant C in (2.104). Henceforth, assume that such a choice has been made. Then a combination of (3.33), (3.35) and (2.105) yields (3.32).

Next, we use (3.28) and (3.32) in order to estimate

$$
\begin{aligned}
I_1 &\le \frac{C}{\sigma(Q)}\,2^{l(d+\gamma)}\,2^{(k+k_0-l)\varepsilon\gamma}\,2^{-k\gamma}\,\mathbf{1}_{\{\rho_\#(w,\cdot)\le\widetilde{C}2^{-l}\}}(z)\int_{y\in E,\,\rho_\#(y,x_Q)\le C_0 2^{(k+k_0-l)\varepsilon}\ell(Q)} 1\,d\sigma(y)\\
&\le C\,2^{-(k+k_0-l)[\gamma-\varepsilon(d+\gamma)]}2^{dl}\,\mathbf{1}_{\{\rho_\#(w,\cdot)\le\widetilde{C}2^{-l}\}}(z),\qquad \forall\,z\in E,
\end{aligned}
\tag{3.36}
$$

where for the last inequality in (3.36) we have used the fact that $(E,\rho|_E,\sigma)$ is a d-dimensional ADR space. At this point we choose $0<\varepsilon<\frac{\gamma}{d+\gamma}<1$ which ensures that $\beta_1:=\gamma-\varepsilon(d+\gamma)>0$, hence

$$
I_1\le C\,2^{dl}\,2^{-(k+k_0-l)\beta_1}\,\mathbf{1}_{\{\rho_\#(w,\cdot)\le\widetilde{C}2^{-l}\}}(z),\qquad \forall\,z\in E. \tag{3.37}
$$

To estimate the contribution from I_1 in the context of (3.25), based on (3.37) and (3.24) we write (recall that $\beta_1=\gamma-\varepsilon(d+\gamma)>0$ is a fixed constant)

$$
\begin{aligned}
\int_E I_1|g(z)|\,d\sigma(z) &\le C\,2^{dl}\,2^{-(k+k_0-l)\beta_1}\int_E |g(z)|\mathbf{1}_{\{\rho_\#(w,\cdot)\le C2^{-l}\}}(z)\,d\sigma(z)\\
&= C2^{-(k+k_0-l)\beta_1}\fint_{z\in E,\,\rho_\#(z,w)\le C2^{-l}}|g(z)|\,d\sigma(z)\\
&= C2^{-(k+k_0-l)\beta_1}(M_E g)(w)\\
&\approx 2^{-|k-l|\beta_1}(M_E g)(w),\quad \text{uniformly in } w\in Q.
\end{aligned}
\tag{3.38}
$$

Next, we turn our attention to I_2 from (3.31). Note that since we are currently assuming that $k+k_0 \geq l$, the condition $\rho_{\#}(y,x_Q) \geq C_0 2^{(k+k_0-l)\varepsilon}\ell(Q)$ forces $y \notin c_1 Q$ for some finite positive constant c_1, which may be further increased as desired by suitably increasing the value of C_0. Thus, assuming that C_0 is sufficiently large to guarantee $c_1 > 1$, we obtain $\mathbf{1}_Q(y) = 0$ if $y \in E$ and $\rho_{\#}(y,x_Q) \geq C_0 2^{(k+k_0-l)\varepsilon}\ell(Q)$. In turn, this implies that $\Phi(x,y) = \delta_E(x)^{\upsilon}\theta(x,y)$ on the domain of integration in I_2. Thus, for each $z \in E$, we have

$$
\begin{aligned}
I_2 \;\leq\; & C2^{-k\upsilon} \int\limits_{y\in E,\, \rho_{\#}(y,x_Q)>C_0 2^{(k+k_0-l)\varepsilon}\ell(Q)} |\theta(x,y)|\,|h_l(y,z)|\,d\sigma(y) \\
& + C2^{-k\upsilon}|h_l(w,z)| \int\limits_{y\in E,\, \rho_{\#}(y,x_Q)>C_0 2^{(k+k_0-l)\varepsilon}\ell(Q)} |\theta(x,y)|\,d\sigma(y) =: I_3 + I_4.
\end{aligned}
\tag{3.39}
$$

We also remark that the design of $\mathcal{U}_Q$ and the fact that $k+k_0-l \geq 0$ ensure that

$$
y \in E,\ \rho_{\#}(y,x_Q) > C_0 2^{(k+k_0-l)\varepsilon}\ell(Q) \Longrightarrow \left\{ \begin{array}{l} \rho_{\#}(x,y) \approx \rho_{\#}(w,y) \approx \rho_{\#}(x_Q,y), \\ \text{uniformly for } x \in \mathcal{U}_Q \text{ and } w \in Q. \end{array} \right.
\tag{3.40}
$$

Making first use of (3.2) combined with (3.40) and the fact that since $x \in \mathcal{U}_Q$ we have $\delta_E(x) \approx \ell(Q) \approx 2^{-k}$, and then of (2.104), we may further estimate

$$
\begin{aligned}
I_3 \;\leq\; & C2^{-k\upsilon} \int\limits_{y\in E,\, \rho_{\#}(y,w)>C2^{(k+k_0-l)\varepsilon}\ell(Q)} \frac{\delta_E(x)^{-a}}{\rho_{\#}(y,w)^{d+\upsilon-a}}\,|h_l(y,z)|\,d\sigma(y) \\
\leq\; & C2^{-k\upsilon}2^{dl} \int\limits_{y\in E,\, \rho_{\#}(y,w)>C2^{(k+k_0-l)\varepsilon}\ell(Q)} \frac{2^{ak}}{\rho_{\#}(y,w)^{d+\upsilon-a}}\,\mathbf{1}_{\{\rho_{\#}(y,\cdot)\leq C2^{-l}\}}(z)\,d\sigma(y) \\
=\; & C\,2^{-(k+k_0-l)\varepsilon(\upsilon-a)}2^{dl} \int_{y\in E,\, \rho_{\#}(y,w)\geq Cr} \frac{r^{\upsilon-a}}{\rho_{\#}(y,w)^{d+\upsilon-a}}\,\mathbf{1}_{\{\rho_{\#}(y,\cdot)\leq C2^{-l}\}}(z)\,d\sigma(y),
\end{aligned}
\tag{3.41}
$$

for each $z \in E$, where we have set

$$
r := 2^{(k+k_0-l)\varepsilon-k}.
\tag{3.42}
$$

Consequently, by (3.41), Fubini's theorem, (3.20) and (3.24), we obtain

$$
\begin{aligned}
\int_E I_3|g(z)|\,d\sigma(z) \leq\; & C\,2^{-(k+k_0-l)\varepsilon(\upsilon-a)} \int_E |g(z)| \times \\
& \times \int_{y\in E,\, \rho_{\#}(y,w)\geq Cr} \frac{r^{\upsilon-a}}{\rho_{\#}(y,w)^{d+\upsilon-a}}\,2^{dl}\,\mathbf{1}_{\{\rho_{\#}(y,\cdot)\leq C2^{-l}\}}(z)\,d\sigma(y)\,d\sigma(z) \\
\leq\; & C\,2^{-(k+k_0-l)\varepsilon(\upsilon-a)} \int_{y\in E,\, \rho_{\#}(y,w)\geq Cr} \frac{r^{\upsilon-a}}{\rho_{\#}(y,w)^{d+\upsilon-a}}\,(M_E g)(y)\,d\sigma(y) \\
\leq\; & C2^{-(k+k_0-l)\varepsilon(\upsilon-a)}(M_E^2 g)(w) \\
\approx\; & 2^{-|k-l|\varepsilon(\upsilon-a)}(M_E^2 g)(w), \quad \text{uniformly in } w \in Q.
\end{aligned}
\tag{3.43}
$$

As for I_4, invoking again (3.2), (3.40), the fact that $\delta_E(x) \approx 2^{-k}$, and (2.104) we write

$$
\begin{aligned}
I_4 &\leq C2^{-k\upsilon}2^{dl}\mathbf{1}_{\{\rho_\#(w,\cdot)\leq C2^{-l}\}}(z) \int_{y\in E,\,\rho_\#(y,x_Q)>Cr} \frac{2^{ak}}{\rho_\#(y,x_Q)^{d+\upsilon-a}}\,d\sigma(y) \\
&= C2^{-(k+k_0-l)\varepsilon(\upsilon-a)}\,2^{dl}\mathbf{1}_{\{\rho_\#(w,\cdot)\leq C2^{-l}\}}(z) \int_{y\in E,\,\rho_\#(y,x_Q)>Cr} \frac{r^{\upsilon-a}}{\rho_\#(y,x_Q)^{d+\upsilon-a}}\,d\sigma(y) \\
&= C2^{-(k+k_0-l)\varepsilon(\upsilon-a)}\,2^{dl}\mathbf{1}_{\{\rho_\#(w,\cdot)\leq C2^{-l}\}}(z), \qquad \forall\, z\in E, \qquad (3.44)
\end{aligned}
$$

by (3.20) (used with $f \equiv 1$). Finally, based on (3.44) and (3.24), we obtain

$$
\begin{aligned}
\int_E I_4|g(z)|\,d\sigma(z) &\leq C\,2^{-(k+k_0-l)\varepsilon(\upsilon-a)}\,2^{dl}\int_E |g(z)|\mathbf{1}_{\{\rho_\#(w,\cdot)\leq C2^{-l}\}}(z)\,d\sigma(z) \\
&\leq C\,2^{-(k+k_0-l)\varepsilon(\upsilon-a)}\,(M_E g)(w) \\
&\approx 2^{-|k-l|\varepsilon(\upsilon-a)}(M_E g)(w), \quad \text{uniformly in } w\in Q. \qquad (3.45)
\end{aligned}
$$

Collectively, (3.41) and (3.44) yield an estimate for I_2, in view of (3.39). In order to express this estimate as well as (3.38) in a manner consistent with (3.23) requires an extra adjustment. Concretely, as a consequence of Lebesgue's Differentiation Theorem (which holds in our context given that σ is Borel regular), the monotonicity of the Hardy-Littlewood maximal operator and Hölder's inequality, for every $r \in [1,\infty)$ and any g locally integrable on E we have

$$
M_E g \leq \big[M_E^2(|g|^r)\big]^{\frac{1}{r}} \quad \text{and} \quad M_E^2 g \leq \big[M_E^2(|g|^r)\big]^{\frac{1}{r}} \qquad \text{pointwise in } E. \qquad (3.46)
$$

Thus, if we define $\beta_2 := \min\{\beta_1, \varepsilon(\upsilon-a)\} > 0$, then a combination of (3.25), (3.31), (3.38), (3.39), (3.43), (3.45), and (3.46) proves (3.23) with β replaced by β_2 in *Case I.*

Case II: $k+k_0<l$. In this scenario, in order to obtain estimate (3.23) we shall use the cancellation property of the operator D_l, and the Hölder regularity of $\theta(\cdot,\cdot)$ in the second variable. To get started, Fubini's Theorem allows us to write

$$
\begin{aligned}
\delta_E(x)^\upsilon\Theta(D_l g)(x) &= \int_E \delta_E(x)^\upsilon \int_E \theta(x,y)h_l(y,z)\,d\sigma(y)\,g(z)\,d\sigma(z) \\
&= \int_E \Psi(x,z)\,g(z)\,d\sigma(z), \qquad \forall\, x\in\mathcal{U}_Q, \qquad (3.47)
\end{aligned}
$$

where we have set

$$
\Psi(x,z) := \delta_E(x)^\upsilon \int_E \theta(x,y)h_l(y,z)\,d\sigma(y), \qquad \forall\, x\in\mathscr{X}\setminus E, \quad \forall\, z\in E. \qquad (3.48)
$$

To proceed, fix $x \in \mathcal{U}_Q$ arbitrary. Based on (2.106) and (2.104), we have

$$
\begin{aligned}
|\Psi(x,z)| &= \left|\delta_E(x)^\upsilon \int_E \big(\theta(x,y)-\theta(x,z)\big)h_l(y,z)\,d\sigma(y)\right| \\
&\leq \delta_E(x)^\upsilon \int_{y\in E,\,\rho_\#(y,z)\leq C2^{-l}} |\theta(x,y)-\theta(x,z)||h_l(y,z)|\,d\sigma(y), \qquad \forall\, z\in E. \qquad (3.49)
\end{aligned}
$$

As a consequence of (3.3), we have

$$|\theta(x,y)-\theta(x,z)|\leq C\,\frac{\rho(y,z)^{\alpha}\delta_E(x)^{-a-\alpha}}{\rho(x,y)^{d+\upsilon-a}}\qquad\text{if }\; y,z\in E,\;\rho(y,z)<\tfrac{1}{2}\rho(x,y). \tag{3.50}$$

Observe that, since we are currently assuming $k+k_0<l$, if the points $y,z\in E$ are such that $\rho_{\#}(y,z)\leq C2^{-l}$, then

$$\rho(y,z)\leq C\rho_{\#}(y,z)\leq C2^{-l}\leq C2^{-k_0}2^{-k}\leq C2^{-k_0}\delta_E(x)\leq C2^{-k_0}\rho(x,y)<\tfrac{1}{2}\rho(x,y), \tag{3.51}$$

where the last inequality follows by choosing k_0 large. For the remainder of the proof fix such a $k_0\in\mathbb{N}_0$. Then (3.50) holds when y belongs to the domain of integration of the last integral in (3.49). For $w\in Q$ arbitrary we claim that

$$\left.\begin{array}{l}\exists\, C'>0\text{ such that }\;\forall\, z,y\in E\\ \text{satisfying }\rho(y,z)\leq C2^{-l}\end{array}\right\}\;\text{one has}\;\left\{\begin{array}{l}\rho(x,y)\geq C'[2^{-k}+\rho_{\#}(w,z)],\\ \text{uniformly for } x\in\mathcal{U}_Q,\ w\in Q.\end{array}\right. \tag{3.52}$$

Indeed, if y,z are as in the left hand-side of (3.52), then $\rho(x,y)\geq C\delta_E(x)\approx\ell(Q)\approx 2^{-k}$. In addition,

$$\begin{aligned}\rho_{\#}(w,z)\;&\leq\;C\rho(w,z)\leq C(\rho(w,x)+\rho(x,y)+\rho(y,z))\leq C(\ell(Q)+\rho(x,y)+2^{-l})\\ &\leq\;C(2^{-k}+\rho(x,y))\leq C\rho(x,y).\end{aligned} \tag{3.53}$$

This proves (3.52). Combining (3.52), (3.50), (2.104) and (3.49), we obtain

$$\begin{aligned}|\Psi(x,z)|\;&\leq\;C2^{-k\upsilon}\int\limits_{y\in E,\,\rho_{\#}(y,z)\leq C2^{-l}}\frac{2^{k(a+\alpha)}2^{-l\alpha}}{[2^{-k}+\rho_{\#}(w,z)]^{d+\upsilon-a}}\,2^{dl}\,d\sigma(y)\\ &\leq\;C2^{-k\upsilon}\frac{2^{k(a+\alpha)}2^{-l\alpha}}{[2^{-k}+\rho_{\#}(w,z)]^{d+\upsilon-a}}\\ &=\;C2^{-|k-l|\alpha}\frac{2^{-k(\upsilon-a)}}{[2^{-k}+\rho_{\#}(w,z)]^{d+\upsilon-a}},\qquad\forall\, z\in E.\end{aligned} \tag{3.54}$$

For the second inequality in (3.54) we used the fact that $(E,\rho|_E,\sigma)$ is a d-dimensional ADR space, while the last equality is a simple consequence of the fact that $k<l$. Thus, returning with (3.54) in (3.47), then making use of (3.20), and then recalling (3.46), we arrive at the conclusion that for each $r\in[1,\infty)$,

$$\begin{aligned}\big|\delta_E(x)^{\upsilon}\Theta(D_l g)(x)\big|\;&\leq\;C2^{-|k-l|\alpha}\int_E\frac{2^{-k(\upsilon-a)}}{[2^{-k}+\rho_{\#}(w,z)]^{d+\upsilon-a}}|g(z)|\,d\sigma(z)\\ &\leq\;C2^{-|k-l|\alpha}(M_E g)(w)\\ &\leq\;C2^{-|k-l|\alpha}\big[M_E^2(|g|^r)(w)\big]^{\frac{1}{r}},\qquad\text{uniformly for }\;w\in Q.\end{aligned} \tag{3.55}$$

In order to estimate $|\delta_E(x)^{\upsilon}(\Theta 1)(x)A_Q(D_l g)|$, first note that (3.30) holds in this case, so by Fubini's Theorem we have

$$|\delta_E(x)^{\upsilon}(\Theta 1)(x)A_Q(D_l g)|\leq C\left|\int_E\Big\{\tfrac{1}{\sigma(Q)}\int_Q h_l(y,z)\,d\sigma(y)\Big\}g(z)\,d\sigma(z)\right|. \tag{3.56}$$

To continue, for some fixed $\varepsilon \in (0,1)$, define

$$S_Q := \big\{x \in Q : \operatorname{dist}_{\rho_\#}(x, E\setminus Q) \le C2^{-|k-l|\varepsilon}\ell(Q)\big\} \quad \text{and} \quad F_Q := Q \setminus S_Q. \tag{3.57}$$

Also, consider a function

$$\begin{array}{c} \eta_Q : E \to \mathbb{R} \ \text{ such that } \ \operatorname{supp}\eta_Q \subseteq Q, \quad 0 \le \eta_Q \le 1, \quad \eta_Q = 1 \text{ on } F_Q \ \text{ and} \\ \Big|\eta_Q(x) - \eta_Q(y)\Big| \le C\Big(\frac{\rho(x,y)}{2^{-|k-l|\varepsilon}\ell(Q)}\Big)^{\gamma}, \qquad \forall\, x \in E,\ \forall\, y \in E, \end{array} \tag{3.58}$$

for some $\gamma \in (0,1)$. That such a function exists is a consequence of Lemma 3.4. Hence,

$$\begin{aligned} \Big|\fint_Q h_l(y,z)\,d\sigma(y)\Big| \ &\le\ \tfrac{1}{\sigma(Q)}\Big|\int_E \big(\mathbf{1}_Q - \eta_Q(y)\big)h_l(y,z)\,d\sigma(y)\Big| \\ &\quad + \tfrac{1}{\sigma(Q)}\Big|\int_E \eta_Q(y)h_l(y,z)\,d\sigma(y)\Big| =: II_1(z) + II_2(z), \quad \forall\, z \in E. \end{aligned} \tag{3.59}$$

Fix $z \in E$. To estimate $II_2(z)$, recall (2.106), (2.104), (3.58) and the fact that E is Ahlfors-David regular. Based on these, we may write

$$\begin{aligned} II_2(z) \ &=\ \tfrac{1}{\sigma(Q)}\Big|\int_E \big(\eta_Q(y) - \eta_Q(z)\big)h_l(y,z)\,d\sigma(y)\Big| \\ &\le\ \tfrac{1}{\sigma(Q)}\int\limits_{y\in E,\,\rho_\#(y,z)\le C2^{-l}} \big|\eta_Q(y) - \eta_Q(z)\big||h_l(y,z)|\,d\sigma(y) \le \tfrac{1}{\sigma(Q)}\left[\frac{2^{-l}}{2^{-|k-l|\varepsilon}2^{-k}}\right]^{\gamma}. \end{aligned} \tag{3.60}$$

In addition, since whenever $y \in \operatorname{supp}\eta_Q \subseteq Q$ and $\rho_\#(y,z) \le C2^{-l} \le C\ell(Q)$ one necessarily has $\rho_\#(w,z) \le C\ell(Q)$, for all $w \in Q$, it follows that one may strengthen (3.60) to

$$II_2(z) \le C\tfrac{1}{\sigma(Q)}2^{-|k-l|(1-\varepsilon)\gamma}\mathbf{1}_{\{\rho_\#(w,\cdot)\le C\ell(Q)\}}(z), \qquad \text{for all } w \in Q. \tag{3.61}$$

Hence, by also recalling (3.46), for each $r \in [1,\infty)$ we further obtain

$$\begin{aligned} \int_E II_2(z)|g(z)|\,d\sigma(z) \ &\le\ C2^{-|k-l|(1-\varepsilon)\gamma}\tfrac{1}{\sigma(Q)}\int_{z\in E,\,\rho_\#(z,w)\le C\ell(Q)} |g(z)|\,d\sigma(z) \\ &\le\ C2^{-|k-l|(1-\varepsilon)\gamma}(M_E g)(w) \le C2^{-|k-l|(1-\varepsilon)\gamma}\big[M_E^2(|g|^r)(w)\big]^{\frac{1}{r}}, \end{aligned} \tag{3.62}$$

for all $w \in Q$. This bound suits our purposes.

Next, we turn our attention to $II_1(z)$. Pick $r \in (1,\infty)$ and let r' be such that $\frac{1}{r} + \frac{1}{r'} = 1$.

Note that since $\operatorname{supp}(\mathbf{1}_Q - \eta_Q) \subseteq S_Q$ and $0 \leq \mathbf{1}_Q - \eta_Q \leq 1$ we may write

$$
\begin{aligned}
\int_E II_1(z)|g(z)|\,d\sigma(z) &\leq \tfrac{1}{\sigma(Q)}\int_E\int_{S_Q}|h_l(y,z)|\,d\sigma(y)\,|g(z)|\,d\sigma(z)\\
&= \tfrac{1}{\sigma(Q)}\int_{S_Q}\int_E|g(z)||h_l(y,z)|\,d\sigma(z)\,d\sigma(y)\\
&\leq C\tfrac{1}{\sigma(Q)}\int_{S_Q}\int_{z\in E,\,\rho_\#(y,z)\leq C2^{-l}}2^{\,dl}|g(z)|\,d\sigma(z)\,d\sigma(y)\\
&\leq C\tfrac{1}{\sigma(Q)}\int_Q\mathbf{1}_{S_Q}(y)(M_Eg)(y)\,d\sigma(y)\\
&\leq C\left[\frac{\sigma(S_Q)}{\sigma(Q)}\right]^{\frac{1}{r'}}\left[⨍_Q(M_Eg)^r\,d\sigma\right]^{\frac{1}{r}}\\
&\leq C\left[\frac{\sigma(S_Q)}{\sigma(Q)}\right]^{\frac{1}{r'}}\big[M_E^2(|g|^r)(w)\big]^{\frac{1}{r}}, \qquad \text{for all } w\in Q. \qquad (3.63)
\end{aligned}
$$

The second inequality in (3.63) is based on (2.104), the third is immediate, the fourth uses Hölder's inequality, while the last one is a consequence of (3.46). By virtue of the "thin boundary" property described in item *(8)* of Proposition 2.12, we have

$$
\exists\, c>0 \ \text{ and } \ \exists\, \tau\in(0,1) \ \text{ such that } \ \sigma(S_Q)\leq c2^{-|k-l|\varepsilon\tau}\sigma(Q), \tag{3.64}
$$

which, when used in concert with (3.63), yields

$$
\int_E II_1(z)|g(z)|\,d\sigma(z)\leq C2^{-|k-l|\varepsilon\tau/r'}\big[M_E^2(|g|^r)(w)\big]^{\frac{1}{r}}, \qquad \text{for all } \ w\in Q. \tag{3.65}
$$

Now choose $\beta_3 := \min\{(1-\varepsilon)\gamma, \frac{\varepsilon\tau}{r'}, \alpha\} > 0$. Then (3.23) follows in the current case with β replaced by β_3, by combining (3.55), (3.56), (3.59), (3.62) and (3.65).

Now the proof of the claim made in Step I is completed by combining what we proved in *Case I* and *Case II* above and by taking $\beta := \min\{\beta_2, \beta_3\} > 0$.

`Step II.` *We claim that there exists a finite constant $C>0$ with the property that for every $f\in L^2(E,\sigma)$ there holds*

$$
\sum_{k\in\mathbb{Z},\,k\geq\kappa_E}\ \sum_{Q\in\mathbb{D}_k(E)}\int_{\mathcal{U}_Q}\big|\delta_E(x)^\upsilon\big((\Theta f)(x)-(\Theta 1)(x)A_Qf\big)\big|^2\frac{d\mu(x)}{\delta_E(x)^{m-d}}\leq C\int_E|f|^2\,d\sigma. \tag{3.66}
$$

To justify this claim, fix $r\in(1,2)$ and let $\beta>0$ be such that (3.23) holds. Then, for an arbitrary function $f\in L^2(E,\sigma)$, using (2.82), we may write

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z},\,k\geq\kappa_E}\ \sum_{Q\in\mathbb{D}_k(E)}\int_{\mathcal{U}_Q}\big|\delta_E(x)^\upsilon\big(\Theta-(\Theta 1)(x)A_Q\big)(f)(x)\big|^2\frac{d\mu(x)}{\delta_E(x)^{m-d}} \qquad (3.67)\\
&\leq 2\sum_{k\in\mathbb{Z},\,k\geq\kappa_E}\ \sum_{Q\in\mathbb{D}_k(E)}\int_{\mathcal{U}_Q}\Big|\sum_{l\in\mathbb{Z},\,l\geq\kappa_E}\delta_E(x)^\upsilon\big(\Theta-(\Theta 1)(x)A_Q\big)(D_l\widetilde{D}_lf)(x)\Big|^2\frac{d\mu(x)}{\delta_E(x)^{m-d}}\\
&\quad+2\sum_{k\in\mathbb{Z},\,k\geq\kappa_E}\ \sum_{Q\in\mathbb{D}_k(E)}\int_{\mathcal{U}_Q}\Big|\delta_E(x)^\upsilon\big(\Theta-(\Theta 1)(x)A_Q\big)(\mathcal{S}_{\kappa_E}(Rf))(x)\Big|^2\frac{d\mu(x)}{\delta_E(x)^{m-d}}=:A_1+A_2.
\end{aligned}
$$

Pick now $\varepsilon \in (0, \beta)$ arbitrary and proceed to estimate A_1 as follows:

$$
\begin{aligned}
A_1 &= 2 \sum_{\substack{k\in\mathbb{Z}\\ k\geq\kappa_E}} \sum_{Q\in\mathbb{D}_k(E)} \int_{\mathcal{U}_Q} \Big| \sum_{\substack{l\in\mathbb{Z}\\ l\geq\kappa_E}} 2^{-|k-l|\varepsilon} 2^{|k-l|\varepsilon} \delta_E(x)^\upsilon \big(\Theta - (\Theta 1)(x) A_Q\big)(D_l \widetilde{D}_l f)(x) \Big|^2 \frac{d\mu(x)}{\delta_E(x)^{m-d}} \\
&\leq 2 \sum_{\substack{k\in\mathbb{Z}\\ k\geq\kappa_E}} \sum_{Q\in\mathbb{D}_k(E)} \int_{\mathcal{U}_Q} \Big(\sum_{l\in\mathbb{Z}} 2^{-2|k-l|\varepsilon} \Big) \times \\
&\qquad\qquad \times \Big(\sum_{\substack{l\in\mathbb{Z}\\ l\geq\kappa_E}} 2^{2|k-l|\varepsilon} \big| \delta_E(x)^\upsilon \big(\Theta - (\Theta 1)(x) A_Q\big)(D_l \widetilde{D}_l f)(x) \big|^2 \Big) \frac{d\mu(x)}{\delta_E(x)^{m-d}} \\
&\leq C \sum_{\substack{l\in\mathbb{Z}\\ l\geq\kappa_E}} \sum_{\substack{k\in\mathbb{Z}\\ k\geq\kappa_E}} \sum_{Q\in\mathbb{D}_k(E)} 2^{2|k-l|\varepsilon} \int_{\mathcal{U}_Q} \big| \delta_E(x)^\upsilon \big(\Theta - (\Theta 1)(x) A_Q\big)(D_l \widetilde{D}_l f)(x) \big|^2 \frac{d\mu(x)}{\delta_E(x)^{m-d}} \\
&\leq C \sum_{\substack{l\in\mathbb{Z}\\ l\geq\kappa_E}} \sum_{\substack{k\in\mathbb{Z}\\ k\geq\kappa_E}} \sum_{Q\in\mathbb{D}_k(E)} 2^{2|k-l|\varepsilon} 2^{-2|k-l|\beta} \inf_{w\in Q} \Big[M_E^2 \big(|\widetilde{D}_l f|^r \big)(w) \Big]^{\frac{2}{r}} \int_{\mathcal{U}_Q} 2^{k(m-d)}\, d\mu \\
&\leq C \sum_{\substack{l\in\mathbb{Z}\\ l\geq\kappa_E}} \sum_{\substack{k\in\mathbb{Z}\\ k\geq\kappa_E}} \sum_{Q\in\mathbb{D}_k(E)} 2^{-2|k-l|(\beta-\varepsilon)} \int_Q \Big[M_E^2 \big(|\widetilde{D}_l f|^r \big) \Big]^{\frac{2}{r}} d\sigma \\
&= C \sum_{\substack{l\in\mathbb{Z}\\ l\geq\kappa_E}} \sum_{\substack{k\in\mathbb{Z}\\ k\geq\kappa_E}} 2^{-2|k-l|(\beta-\varepsilon)} \int_E \Big[M_E^2 \big(|\widetilde{D}_l f|^r \big) \Big]^{\frac{2}{r}} d\sigma \leq C \sum_{\substack{l\in\mathbb{Z}\\ l\geq\kappa_E}} \int_E \Big[M_E^2 \big(|\widetilde{D}_l f|^r \big) \Big]^{\frac{2}{r}} d\sigma \\
&\leq C \sum_{\substack{l\in\mathbb{Z}\\ l\geq\kappa_E}} \int_E |\widetilde{D}_l f|^2\, d\sigma \leq C \int_E |f|^2\, d\sigma. \qquad (3.68)
\end{aligned}
$$

The first inequality in (3.68) uses the Cauchy-Schwarz inequality, the second inequality uses the fact that $\sum\limits_{l\in\mathbb{Z}} 2^{-2|k-l|\varepsilon} = C$, for some finite positive constant independent of $k \in \mathbb{Z}$, the third inequality employs (3.23), while the forth inequality is based on the fact that $\mu(\mathcal{U}_Q) \leq C 2^{-km}$ and $2^{-kd} \leq C\sigma(Q)$, for all $Q \in \mathbb{D}_k(E)$. Since $\varepsilon \in (0, \beta)$, we have $\sum\limits_{k\in\mathbb{Z}} 2^{-2|k-l|(\beta-\varepsilon)} = C$, which is used in the fifth inequality in (3.68). The sixth inequality in (3.68) follows from the boundedness of the Hardy-Littlewood maximal operator M_E on $L^{\frac{2}{r}}(E, \sigma)$ (recall that $2/r > 1$) and the fact that $|\widetilde{D}_l f|^r \in L^{\frac{2}{r}}(E, \sigma)$. Finally, the last inequality in (3.68) uses (2.81).

There remains to obtain a similar bound for A_2 introduced in (3.67). Note that if E is unbounded then $\kappa_E = -\infty$ so actually $A_2 = 0$ since we agreed that $\mathcal{S}_{-\infty} = 0$. Consider therefore the case when E is bounded, in which scenario we have $k \geq \kappa_E \in \mathbb{Z}$. An inspection of the proof of (3.23) in *Case I* (of Step I) shows that the function $D_l g = \mathcal{S}_{l+1} g - \mathcal{S}_l g$ may actually be decoupled, i.e., be replaced by, say, $\mathcal{S}_l g$. This is because in *Case I* we have only made use of the regularity of the integral kernel of D_l (as opposed to *Case II* where the vanishing condition of the integral kernel of D_l is used) and the integral kernel of $\mathcal{S}_l$ exhibits the same type of regularity. Consequently, the same proof as before gives that for every $r \in (1, \infty)$ there exist finite positive constants C and β such that for each $k, l \in \mathbb{Z}$ with $k, l \geq \kappa_E$ such that $k \geq l$ and

every $Q \in \mathbb{D}_k(E)$, there holds

$$\sup_{x\in\mathcal{U}_Q}\Big|\delta_E(x)^{\upsilon}\big(\Theta(\mathcal{S}_l g)(x)-(\Theta 1)(x)A_Q(\mathcal{S}_l g)\big)\Big| \leq C2^{-|k-l|\beta}\inf_{w\in Q}\Big[M_E^2(|g|^r)(w)\Big]^{\frac{1}{r}}, \tag{3.69}$$

for every $g : E \to \mathbb{R}$ locally integrable. Applying (3.69) with $l := \kappa_E$ and $g := Rf$ then yields

$$\begin{aligned}
A_2 &\leq C\sum_{k\in\mathbb{Z},\,k\geq\kappa_E}\ \sum_{Q\in\mathbb{D}_k(E)} 2^{-2|k-\kappa_E|\beta}\inf_{w\in Q}\Big[M_E^2\big(|Rf|^r\big)(w)\Big]^{\frac{2}{r}}\int_{\mathcal{U}_Q}2^{k(m-d)}\,d\mu \\
&\leq C\sum_{\substack{k\in\mathbb{Z}\\ k\geq\kappa_E}}\ \sum_{Q\in\mathbb{D}_k(E)} 2^{-2|k-\kappa_E|\beta}\int_Q\Big[M_E^2\big(|Rf|^r\big)\Big]^{\frac{2}{r}}\,d\sigma \\
&= C\sum_{\substack{k\in\mathbb{Z}\\ k\geq\kappa_E}} 2^{-2|k-\kappa_E|\beta}\int_E\Big[M_E^2\big(|Rf|^r\big)\Big]^{\frac{2}{r}}\,d\sigma \leq C\int_E\Big[M_E^2\big(|Rf|^r\big)\Big]^{\frac{2}{r}}\,d\sigma \\
&\leq C\int_E|Rf|^2\,d\sigma \leq C\int_E|f|^2\,d\sigma,
\end{aligned} \tag{3.70}$$

since R is a bounded operator on $L^2(E,\sigma)$. Now (3.67), (3.68) and (3.70) imply (3.66) completing the proof of the claim made in Step II.

`Step III.` *The end-game in the proof of the implication "(3.5)⇒(3.6)".* Fix $f \in L^2(E,\sigma)$ and recall $\epsilon \in (0,1)$ from Lemma 2.21 (here is where we use that C_* is as in (2.132)). Then by (2.133) and (2.158) we may write

$$\begin{aligned}
&\int_{\{x\in\mathscr{X}\setminus E:\,\delta_E(x)<\epsilon\,\mathrm{diam}_\rho(E)\}}|(\Theta f)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x) \\
&\qquad\leq\int_{\bigcup\limits_{Q\in\mathbb{D}(E)}\mathcal{U}_Q}|(\Theta f)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x) \\
&\qquad\leq C\sum_{k\in\mathbb{Z},\,k\geq\kappa_E}\ \sum_{Q\in\mathbb{D}_k(E)}\int_{\mathcal{U}_Q}\big|(\Theta f)(x)-(\Theta 1)(x)A_Q f\big|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x) \\
&\qquad\quad+C\sum_{k\in\mathbb{Z},\,k\geq\kappa_E}\ \sum_{Q\in\mathbb{D}_k(E)}\int_{\mathcal{U}_Q}\big|(\Theta 1)(x)A_Q f\big|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x).
\end{aligned} \tag{3.71}$$

Observe that if we set $B_Q := \int_{\mathcal{U}_Q}|(\Theta 1)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)$ for each $Q \in \mathbb{D}(E)$, then in view of (2.131), (2.158), and (3.5) there holds

$$\sum_{Q'\in\mathbb{D}(E),\,Q'\subseteq Q} B_Q \leq C\int_{T_E(Q)}|(\Theta 1)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x) \leq C\sigma(Q), \quad \forall\, Q \in \mathbb{D}(E). \tag{3.72}$$

Thus, the numerical sequence $\{B_Q\}_{Q\in\mathbb{D}(E)}$ satisfies (3.8). Consequently, Lemma 3.3 applies

and gives

$$\sum_{Q\in\mathbb{D}(E)}\int_{\mathcal{U}_Q}\big|(\Theta 1)(x)A_Q f\big|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)$$
$$\leq C\int_E\Big[\sup_{Q\in\mathbb{D}(E),\,x\in Q}\fint_Q|f|\,d\sigma\Big]^2\,d\sigma(x)$$
$$\leq C\int_E(M_E f)^2\,d\sigma\leq C\int_E|f|^2\,d\sigma,\tag{3.73}$$

where for the last inequality in (3.73) we have used the boundedness of M_E on $L^2(E,\sigma)$. By combining (3.71), (3.66) and (3.73) we therefore obtain

$$\int_{\{x\in\mathscr{X}\setminus E:\,\delta_E(x)<\epsilon\,\mathrm{diam}_\rho(E)\}}|(\Theta f)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\leq C\int_E|f|^2\,d\sigma.\tag{3.74}$$

Of course, this takes care of (3.6) in the case when $\mathrm{diam}_\rho(E)=\infty$. To prove that (3.6) continues to hold in the case when E is bounded, let $R:=\mathrm{diam}_\rho(E)\in(0,\infty)$, fix $x_0\in E$ and set $\mathcal{O}:=\big\{x\in\mathscr{X}\setminus E:\,\epsilon R\leq\delta_E(x)\big\}$. Then for each $x\in\mathcal{O}$ there exists $y\in E$ such that $\rho_\#(x,y)<2\delta_E(x)$, hence

$$\rho(x,x_0)\leq C_\rho^2\rho_\#(x,x_0)\leq C_\rho^2\max\{\rho_\#(x,y),\rho_\#(y,x_0)\}\leq C_\rho^2\max\{2,\tfrac{1}{\epsilon}\}\delta_E(x).\tag{3.75}$$

Thus, $\rho(x,x_0)\approx\delta_E(x)$ uniformly for $x\in\mathcal{O}$. Based on this, estimate (3.2), and Hölder's inequality we then obtain the pointwise estimate $|(\Theta f)(x)|^2\leq CR^d\|f\|^2_{L^2(E,\sigma)}\rho(x,x_0)^{-2(d+\upsilon)}$ for each $x\in\mathcal{O}$. Consequently, for some sufficiently small $c>0$ and some $C\in(0,\infty)$ independent of f and R, we may estimate

$$\int_{\mathcal{O}}|(\Theta f)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)$$
$$\leq CR^d\|f\|^2_{L^2(E,\sigma)}\int_{\mathscr{X}\setminus B_{\rho_\#}(x_0,cR)}\rho_\#(x,x_0)^{-m-d}\,d\mu(x)$$
$$\leq CR^d\|f\|^2_{L^2(E,\sigma)}\sum_{j=1}^\infty\int_{B_{\rho_\#}(x_0,c2^{j+1}R)\setminus B_{\rho_\#}(x_0,c2^jR)}\rho_\#(x,x_0)^{-m-d}\,d\mu(x)$$
$$\leq CR^d\|f\|^2_{L^2(E,\sigma)}\sum_{j=1}^\infty(2^jR)^{-m-d}\mu\big(B_{\rho_\#}(x_0,c2^{j+1}R)\big)$$
$$\leq CR^d\|f\|^2_{L^2(E,\sigma)}\sum_{j=1}^\infty(2^jR)^{-m-d}(2^jR)^m\leq C\|f\|^2_{L^2(E,\sigma)}.\tag{3.76}$$

Now (3.6) follows by combining (3.74) and (3.76).

At this stage in the proof of the theorem, we are left with establishing the converse implication with the regularity assumption (3.3) on the kernel now dropped. With the goal of proving (3.5), suppose that (3.7) holds for some $\eta\in(0,\infty)$. Assume first that $\mathrm{diam}_\rho(E)<\infty$

and pick an arbitrary $\eta_o \in (\eta, \infty)$. We may then estimate

$$
\begin{aligned}
\int_{\left\{x\in\mathscr{X}\setminus E:\,\delta_E(x)<\eta_o\,\mathrm{diam}_\rho(E)\right\}} & |(\Theta f)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x) \\
&= \int_{\left\{x\in\mathscr{X}\setminus E:\,\delta_E(x)<\eta\,\mathrm{diam}_\rho(E)\right\}} |(\Theta f)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x) \qquad (3.77)\\
&\quad + \int_{\left\{x\in\mathscr{X}\setminus E:\,\eta\,\mathrm{diam}_\rho(E)\leq\delta_E(x)<\eta_o\,\mathrm{diam}_\rho(E)\right\}} |(\Theta f)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x),
\end{aligned}
$$

and then observing that since by (3.2) and Hölder's inequality we have the pointwise estimate $|(\Theta f)(x)|^2 \leq C\|f\|^2_{L^2(E,\sigma)}[\mathrm{diam}_\rho(E)]^{-d-2\upsilon}$ whenever $\eta\,\mathrm{diam}_\rho(E) \leq \delta_E(x) < \eta_o\,\mathrm{diam}_\rho(E)$, the last integral in (3.77) may also be bounded by $C\int_E |f|^2\,d\sigma$, for some finite positive geometric constant independent of $\mathrm{diam}_\rho(E)$. The bottom line is that there is no loss of generality in assuming that $\eta > 0$ appearing in (3.7) is as large as desired.

Assuming that this is the case, fix $Q \in \mathbb{D}(E)$ and, for some large finite positive constant C_o, write

$$
\begin{aligned}
\tfrac{1}{\sigma(Q)}\int_{T_E(Q)} & |\Theta 1(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x) \\
&\leq \tfrac{2}{\sigma(Q)}\int_{T_E(Q)}\big|\big(\Theta\mathbf{1}_{E\cap B_{\rho_\#}(x_Q,C_o\ell(Q))}\big)(x)\big|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x) \\
&\quad + \tfrac{2}{\sigma(Q)}\int_{T_E(Q)}\big|\big(\Theta\mathbf{1}_{E\setminus B_{\rho_\#}(x_Q,C_o\ell(Q))}\big)(x)\big|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x) \\
&=: \mathcal{I}_1 + \mathcal{I}_2. \qquad (3.78)
\end{aligned}
$$

Then, granted (3.7) and keeping in mind (2.145) and the fact that η is large, we may write

$$
\begin{aligned}
\mathcal{I}_1 &\leq \tfrac{2}{\sigma(Q)}\int_{\{x\in\mathscr{X}:\,0<\delta_E(x)<\eta\,\mathrm{diam}_\rho(E)\}}\big|\big(\Theta\mathbf{1}_{E\cap B_{\rho_\#}(x_Q,C_o\ell(Q))}\big)(x)\big|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x) \\
&\leq \tfrac{C}{\sigma(Q)}\int_E\big|\mathbf{1}_{E\cap B_{\rho_\#}(x_Q,C_o\ell(Q))}(x)\big|^2\,d\sigma(x) \\
&\leq C\frac{\sigma\big(E\cap B_{\rho_\#}(x_Q,C_o\ell(Q))\big)}{\sigma(Q)} \leq C, \qquad (3.79)
\end{aligned}
$$

given that σ is doubling. To estimate $\mathcal{I}_2$, observe that there exists $C \in (0,\infty)$ with the property that

$$
\begin{aligned}
&\big|\big(\Theta\mathbf{1}_{E\setminus B_{\rho_\#}(x_Q,C_o\ell(Q))}\big)(x)\big| \qquad (3.80)\\
&\quad \leq C\int_{E\setminus B_{\rho_\#}(x_Q,C_o\ell(Q))}\frac{\delta_E(x)^{-a}}{\rho_\#(x,y)^{d+\upsilon-a}}\,d\sigma(y) \leq C\ell(Q)^{-(\upsilon-a)}\delta_E(x)^{-a}, \qquad \forall\, x\in T_E(Q).
\end{aligned}
$$

This is based on (3.2), (3.20) (used here with $f \equiv 1$ and $\varepsilon = \upsilon - a > 0$) and on the fact that C_o has been chosen sufficiently large (compare with (3.40)). Consequently, from (3.80), (2.145),

and (3.22) in Lemma 3.6 (for which we recall that $\upsilon - a > 0$)

$$\begin{aligned}
\mathcal{I}_2 &\leq \tfrac{C}{\sigma(Q)}\ell(Q)^{-2(\upsilon-a)}\int_{T_E(Q)}\delta_E(x)^{2(\upsilon-a)-(m-d)}\,d\mu(x)\\
&\leq \tfrac{C}{\sigma(Q)}\ell(Q)^{-2(\upsilon-a)}\int_{B_{\rho_\#}(x_Q,C\ell(Q))}\delta_E(x)^{2(\upsilon-a)-(m-d)}\,d\mu(x)\\
&\leq \tfrac{C}{\sigma(Q)}\ell(Q)^{-2(\upsilon-a)}\ell(Q)^{m-d+2(\upsilon-a)-(m-d)}\ell(Q)^d \leq C < \infty, \qquad (3.81)
\end{aligned}$$

given that $\sigma(Q) \approx \ell(Q)^d$. In concert, (3.78)-(3.81) prove (3.5), and this finishes the proof of the theorem. $\square$

3.2 An arbitrary codimension local $T(b)$ theorem for square functions

We continue to work in the context introduced at the beginning of Section 3. The main result in this subsection is a local $T(b)$ theorem for square functions, to the effect that *a square function estimate for the integral operator Θ holds if there exists a suitably nondegenerate family of functions $\{b_Q\}$, indexed by dyadic cubes Q in E, for which there is local scale-invariant L^2 control of Θb_Q, appropriately weighted by a power of the distance to E*. To state this formally, the reader is again advised to recall the dyadic cube grid from Proposition 2.12 and the regularized distance function to a set from (2.19).

Theorem 3.7. *Let d, m be two real numbers such that $0 < d < m$. Assume that $(\mathscr{X}, \rho, \mu)$ is an m-dimensional* ADR *space, E is a closed subset of $(\mathscr{X}, \tau_\rho)$, and σ is a Borel regular measure on $(E, \tau_{\rho|_E})$ with the property that $(E, \rho\big|_E, \sigma)$ is a d-dimensional* ADR *space.*

Suppose that Θ is the integral operator defined in (3.4) *with a kernel θ as in* (3.1), (3.2), (3.3). *Furthermore, let $\mathbb{D}(E)$ denote a dyadic cube structure on E, consider a Whitney covering $\mathbb{W}_\lambda(\mathscr{X} \setminus E)$ of $\mathscr{X} \setminus E$ and a constant C_* as in Lemma 2.21 and, corresponding to these, recall the dyadic Carleson tents from* (2.131).

For these choices, assume that there exist finite constant $C_0 \geq 1$, $c_0 \in (0, 1]$, and a collection $\{b_Q\}_{Q\in\mathbb{D}(E)}$ of σ-measurable functions $b_Q : E \to \mathbb{C}$ such that for each $Q \in \mathbb{D}(E)$ the following estimates hold:

1. $\int_E |b_Q|^2\,d\sigma \leq C_0\sigma(Q)$;
2. *there exists* $\widetilde{Q} \in \mathbb{D}(E)$, $\widetilde{Q} \subseteq Q$, $\ell(\widetilde{Q}) \geq c_0\ell(Q)$, *and* $\left|\int_{\widetilde{Q}} b_Q\,d\sigma\right| \geq \frac{1}{C_0}\,\sigma(\widetilde{Q})$;
3. $\int_{T_E(Q)} |(\Theta\, b_Q)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x) \leq C_0\sigma(Q)$.

Then there exists a finite constant $C > 0$ depending only on C_0, C_θ, and the ADR *constants of E and $\mathscr{X}$, as well as on $\operatorname{diam}_\rho(E)$ in the case when E is bounded, such that for each function $f \in L^2(E, \sigma)$ one has*

$$\int_{\mathscr{X}\setminus E} |(\Theta f)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x) \leq C\int_E |f(x)|^2\,d\sigma(x). \qquad (3.82)$$

Before giving the proof of Theorem 3.7 we present a stopping-time construction and elaborate on the way this is used.

Lemma 3.8. *Assume (E,ρ,σ) is a space of homogeneous type with the property that σ is Borel regular, and denote by $\mathbb{D}(E)$ a dyadic cube structure on E. Suppose that there exist finite constants $C_0 \geq 1$, $c_0 \in (0,1)$, and a collection $\{b_Q\}_{Q\in\mathbb{D}(E)}$ of σ-measurable functions $b_Q : E \to \mathbb{C}$ such that*

$$\int_E |b_Q|^2 \, d\sigma \leq C_0\sigma(Q) \quad \textit{for every } Q \in \mathbb{D}(E), \textit{ and} \tag{3.83}$$

$$\forall\, Q \in \mathbb{D}(E) \quad \exists\, \widetilde{Q} \in \mathbb{D}(E),\, \widetilde{Q} \subseteq Q,\, \ell(\widetilde{Q}) \geq c_0\ell(Q) \textit{ with } \left|\int_{\widetilde{Q}} b_Q \, d\sigma\right| \geq \frac{1}{C_0}\,\sigma(\widetilde{Q}). \tag{3.84}$$

Then there exists a number $\eta \in (0,1)$ such that for every cube $Q \in \mathbb{D}(E)$, and each fixed $\widetilde{Q}$ as in (3.84), one can find a sequence $\{Q_j\}_{j\in J} \subseteq \mathbb{D}(E)$ of pairwise disjoint cubes satisfying the following properties:

(i) $Q_j \subseteq \widetilde{Q}$ for every $j \in J$ and $\sigma\Big(\widetilde{Q} \setminus \bigcup_{j\in J} Q_j\Big) \geq \eta\,\sigma(\widetilde{Q})$;

(ii) if

$$\mathcal{F}_Q := \big\{Q' \in \mathbb{D}(E) : Q' \subseteq \widetilde{Q} \textit{ and } Q' \textit{ is not contained in } Q_j \textit{ for every } j \in J\big\}, \tag{3.85}$$

then $\left|⨍_{Q'} b_Q \, d\sigma\right| \geq \frac{1}{2}$ for every $Q' \in \mathcal{F}_Q$.

Proof. Granted the regularity of the measure σ, it follows from *(3)* and *(9)* in Proposition 2.12 that for each $k \in \mathbb{Z}$ and each $Q \in \mathbb{D}_k(E)$ we have

$$\sigma\Big(Q \setminus \bigcup_{Q'\subseteq Q,\, Q'\in\mathbb{D}_\ell(E)} Q'\Big) = 0 \qquad \text{for every } \ell \in \mathbb{Z} \text{ with } \ell \geq k. \tag{3.86}$$

Thanks to (3.83)-(3.84), we may re-normalize the functions $\{b_Q\}_{Q\in\mathbb{D}(E)}$ so that $⨍_{\widetilde{Q}} b_Q \, d\sigma = 1$ for each $Q \in \mathbb{D}(E)$, where $\widetilde{Q}$ is as in (3.84). In the process, the first inequality in (3.83) becomes

$$\int_E |b_Q|^2 \, d\sigma \leq C_0^3\sigma(Q), \qquad \text{for each} \quad Q \in \mathbb{D}(E). \tag{3.87}$$

Fix $Q \in \mathbb{D}(E)$ and a corresponding $\widetilde{Q}$ as in (3.84). In particular we have

$$\sigma(Q) \leq C_1\sigma(\widetilde{Q}) \quad \text{for some } C_1 \in [1,\infty) \text{ independent of } Q, \widetilde{Q}. \tag{3.88}$$

Next, perform a stopping-time argument for $\widetilde{Q}$ by successively dividing it into dyadic sub-cubes $Q' \subseteq \widetilde{Q}$ and stopping whenever $\mathrm{Re}\,⨍_{Q'} b_Q \, d\sigma \leq \frac{1}{2}$. That this is doable is ensured by (3.86) and the re-normalization of b_Q. This yields a family of cubes $\{Q_j\}_{j\in J} \subseteq \mathbb{D}(E)$ such that:

(1) $Q_j \subseteq \widetilde{Q} \subseteq Q$ for each $j \in J$ and $Q_j \cap Q_{j'} = \emptyset$ whenever $j, j' \in J$, $j \neq j'$;

(2) $\mathrm{Re}\,⨍_{Q_j} b_Q \, d\sigma \leq \frac{1}{2}$ for each $j \in J$;

(3) the family $\{Q_j\}_{j\in J}$ is maximal with respect to (1) and (2) above, i.e., if $Q' \in \mathbb{D}(E)$ is such that $Q' \subseteq \widetilde{Q}$, then either there exists $j_0 \in J$ such that $Q' \subseteq Q_{j_0}$, or $\mathrm{Re}\,⨍_{Q'} b_Q \, d\sigma > \frac{1}{2}$.

Then we may write

$$
\begin{aligned}
\sigma(\widetilde{Q}) &= \int_{\widetilde{Q}} b_Q\,d\sigma = \mathrm{Re}\int_{\widetilde{Q}\setminus(\bigcup_{j\in J}Q_j)} b_Q\,d\sigma + \sum_{j\in J}\mathrm{Re}\int_{Q_j} b_Q\,d\sigma \\
&\le \Big(\int_E |b_Q|^2 d\sigma\Big)^{\frac12}\sigma\big(\widetilde{Q}\setminus\cup_{j\in J}Q_j\big)^{\frac12} + \tfrac12\sum_{j\in J}\sigma(Q_j) \\
&\le C_0^{\frac32}\sigma(Q)^{\frac12}\sigma\big(\widetilde{Q}\setminus\cup_{j\in J}Q_j\big)^{\frac12} + \tfrac12\sigma(\widetilde{Q}) \\
&\le C_1^{\frac12}C_0^{\frac32}\sigma(\widetilde{Q})^{\frac12}\sigma\big(\widetilde{Q}\setminus\cup_{j\in J}Q_j\big)^{\frac12} + \tfrac12\sigma(\widetilde{Q}),
\end{aligned}
\tag{3.89}
$$

where the first inequality in (3.89) is based on Hölder's inequality and condition (2) above, the second inequality uses (3.87) and (1) above, while the last inequality is a consequence of (3.88). After absorbing $\frac12\sigma(\widetilde{Q})$ in the leftmost side of (3.89) and setting $\eta := \frac{1}{4C_1C_0^3}\in(0,1)$, it follows that $\sigma\big(\widetilde{Q}\setminus\cup_{j\in J}Q_j\big)\ge\eta\sigma(\widetilde{Q})$, thus condition (i) holds for the family $\{Q_j\}_{j\in J}$ constructed above. In addition it is immediate from property (3) that condition (ii) is also satisfied. $\square$

A typical application of Lemma (3.8) is exemplified by our next result.

Lemma 3.9. *Assume that* $(\mathscr{X},\rho)$ *is a geometrically doubling quasi-metric space,* μ *is a Borel measure on* $(\mathscr{X},\tau_\rho)$ *and that* E *is a nonempty, closed, proper subset of* $(\mathscr{X},\tau_\rho)$*. Also, suppose that* σ *is a Borel regular measure on* E *with the property that* $(E,\rho\big|_E,\sigma)$ *is a space of homogeneous type and denote by* $\mathbb{D}(E)$ *a dyadic cube structure on* E*. Next, assume that* $\mathbb{W}_\lambda(\mathscr{X}\setminus E)$ *is a Whitney covering of* $\mathscr{X}\setminus E$ *as in Lemma 2.24 (for some fixed* $a\ge1$*), and recall the regions* $\{\mathcal{U}_Q\}_{Q\in\mathbb{D}(E)}$ *from* (2.130) *relative to this cover. Finally, assume the hypotheses of Lemma 3.8, and for each cube* $Q\in\mathbb{D}(E)$*, recall the collection* $\mathcal{F}_Q$ *from* (3.85) *and define*

$$
E_Q^* := \bigcup_{Q'\in\mathcal{F}_Q}\mathcal{U}_{Q'}. \tag{3.90}
$$

Then for every $\gamma\in\mathbb{R}$ *and every* μ*-measurable function* $u:\mathscr{X}\setminus E\to\mathbb{R}$ *it follows that*

$$
\int_{E_Q^*}|u(x)|^2\delta_E(x)^\gamma\,d\mu(x) \approx \sum_{Q'\in\mathcal{F}_Q}\int_{\mathcal{U}_{Q'}}\big|u(x)⨍_{Q'}b_Q\,d\sigma\big|^2\delta_E(x)^\gamma\,d\mu(x), \tag{3.91}
$$

with finite positive equivalence constants, depending only on C_0 *from* (3.83)-(3.84).

Proof. This readily follows by combining (2.158), (3.83)-(3.84) and (ii) in Lemma 3.8. $\square$

We are now ready to present the

Proof of Theorem 3.7. Based on Theorem 3.2, it suffices to show that $|\Theta 1|^2\delta_E^{2\upsilon-(m-d)}\,d\mu$ is a Carleson measure in $\mathscr{X}\setminus E$ relative to E, that is, that (3.5) holds. In a first stage, we shall show that (3.5) holds for Θ replaced by some truncated operators Θ_i, $i\in\mathbb{N}$. More precisely, for each $i\in\mathbb{N}$ consider the kernel

$$
\theta_i(x,y) := \mathbf{1}_{\{1/i<\delta_E<i\}}(x)\theta(x,y), \qquad \forall\,x\in\mathscr{X}\setminus E,\quad \forall\,y\in E, \tag{3.92}
$$

and introduce the integral operator mapping $f : E \to \mathbb{R}$ into

$$(\Theta_i f)(x) := \int_E \theta_i(x,y) f(y)\, d\sigma(y), \qquad \forall\, x \in \mathscr{X} \setminus E. \tag{3.93}$$

Clearly,

$$\Theta_i = \mathbf{1}_{\{1/i<\delta_E<i\}}\Theta, \qquad \forall\, i \in \mathbb{N}, \tag{3.94}$$

and, with C_θ as in (3.2), for every $x \in \mathscr{X} \setminus E$ we have

$$|\theta_i(x,y)| \leq C_\theta \frac{\delta_E(x)^{-a}}{\rho(x,y)^{d+\upsilon-a}}, \qquad \forall\, y \in E, \tag{3.95}$$

$$|\theta_i(x,y) - \theta_i(x,\widetilde{y})| \leq C_\theta \frac{\rho(y,\widetilde{y})^\alpha \delta_E(x)^{-a-\alpha}}{\rho(x,y)^{d+\upsilon-a}}, \qquad \forall\, \widetilde{y}, y \in E, \quad \rho(y,\widetilde{y}) \leq \tfrac{1}{2}\rho(x,y). \tag{3.96}$$

Then for each $i \in \mathbb{N}$ and each $x \in \mathscr{X} \setminus E$ by (3.93), (3.95) and Lemma 3.5 (given that $\upsilon - a > 0$) we have

$$\begin{aligned} |(\Theta_i 1)(x)| &\leq C\mathbf{1}_{\{1/i<\delta_E<i\}}(x) \int_E \frac{\delta_E(x)^{-a}}{\rho_\#(x,y)^{d+\upsilon-a}}\, d\sigma(y) \leq C\mathbf{1}_{\{1/i<\delta_E<i\}}(x)[\delta_E(x)]^{-\upsilon} \\ &\leq Ci^\upsilon \mathbf{1}_{\{1/i<\delta_E<i\}}(x). \end{aligned} \tag{3.97}$$

Recalling now (2.145), estimate (3.97) further yields (with x_Q denoting the center of Q)

$$\begin{aligned} &\int_{T_E(Q)} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\qquad \leq Ci^{2\upsilon} \int_{x \in B_{\rho_\#}(x_Q, C\ell(Q)),\, \delta_E(x)<i} \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\qquad \leq Ci^{4\upsilon}\ell(Q)^d \leq Ci^{4\upsilon}\sigma(Q), \qquad \forall\, Q \in \mathbb{D}(E), \end{aligned} \tag{3.98}$$

for some constant $C \in (0,\infty)$ which does not depend on Q and i, where the second inequality in (3.98) is a consequence of Lemma 3.6. Hence, if we now define

$$c_i := \sup_{Q \in \mathbb{D}(E)} \Big[\tfrac{1}{\sigma(Q)} \int_{T_E(Q)} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x)\Big], \qquad \forall\, i \in \mathbb{N}, \tag{3.99}$$

then (3.98) implies $0 \leq c_i \leq Ci^{4\upsilon}$ for each $i \in \mathbb{N}$. In particular, each c_i is finite. Our goal is to show that actually

$$\sup_{i \in \mathbb{N}} c_i < \infty. \tag{3.100}$$

To this end, fix $Q \in \mathbb{D}(E)$ and for this Q consider some $\widetilde{Q}$ satisfying *2.* in the hypothesis of Theorem 3.7 (recall the constant c_0). In particular, we have that

$$\exists\, p \in \mathbb{N} \quad \text{satisfying} \quad p \leq -\log_2(c_0) \quad \text{and such that} \quad \widetilde{Q} \in \mathbb{D}_p(E). \tag{3.101}$$

Next recall Lemma 3.8 and Lemma 3.9 and the notation therein. Then, from the definition of E_Q^*, (3.101) and (2.131), we have

$$T_E(Q) \subseteq E_Q^* \cup \Big(\bigcup_{j\in J} T_E(Q_j)\Big) \cup \Big(\bigcup_{Q''\in\mathbb{D}_p(E),\, Q''\subseteq Q,\, Q''\neq\widetilde{Q}} T_E(Q'')\Big). \tag{3.102}$$

Consequently, for each $i \in \mathbb{N}$ we may write

$$\begin{aligned}
&\int_{T_E(Q)} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \qquad (3.103)\\
&\quad\le \int_{E_Q^*} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) + \sum_{j\in J} \int_{T_E(Q_j)} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x)\\
&\quad + \sum_{\substack{Q''\in\mathbb{D}_p(E),\, Q''\subseteq Q\\ Q''\neq\widetilde{Q}}} \int_{T_E(Q'')} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x).
\end{aligned}$$

To estimate the first integral in the right hand-side of (3.103) start with (3.91) written for $u := \Theta_i 1$. Keeping in mind that $|\Theta_i 1| \le |\Theta 1|$ for all $i \in \mathbb{N}$, we obtain

$$\begin{aligned}
\int_{E_Q^*} |\Theta_i 1(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) &\le C \sum_{Q'\in\mathcal{F}_Q} \int_{\mathcal{U}_{Q'}} |(\Theta 1)(x) A_{Q'} b_Q|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x)\\
&\le C \sum_{Q'\in\mathbb{D}(E),\, Q'\subseteq Q} \int_{\mathcal{U}_{Q'}} |(\Theta b_Q)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x)\\
&\quad + C \sum_{Q'\in\mathbb{D}(E)} \int_{\mathcal{U}_{Q'}} \big|(\Theta b_Q)(x) - (\Theta 1)(x) A_{Q'} b_Q\big|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x)\\
&\le C \int_{T_E(Q)} |(\Theta b_Q)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) + C \int_E |b_Q|^2\, d\sigma \le C\sigma(Q).
\end{aligned} \tag{3.104}$$

The second inequality in (3.104) is immediate, the third uses (2.131) and (3.66) (the latter applied with $f := b_Q$), while the fourth uses assumptions *1* and *3* of Theorem 3.7.

Consider next the first sum in the right hand-side of (3.103). Upon recalling (3.99) and the properties of Q_j's in Lemma 3.8 we may write

$$\begin{aligned}
\sum_{j\in J} \int_{T_E(Q_j)} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) &\le c_i \sum_{j\in J} \sigma(Q_j) = c_i\, \sigma\big(\cup_{j\in J} Q_j\big)\\
&= c_i\, \sigma(\widetilde{Q}) - c_i\, \sigma\big(\widetilde{Q} \setminus \cup_{j\in J} Q_j\big)\\
&\le c_i\, (1-\eta)\, \sigma(\widetilde{Q}).
\end{aligned} \tag{3.105}$$

Upon recalling (3.99) we obtain

$$\sum_{\substack{Q''\in\mathbb{D}_p(E),\, Q''\subseteq Q\\ Q''\neq\widetilde{Q}}} \int_{T_E(Q'')} |(\Theta_i 1)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \le c_i \sigma(Q \setminus \widetilde{Q}). \tag{3.106}$$

In concert, (3.103), (3.104), (3.105), and (3.106) imply that there exists a finite constant $C>0$ with the property that for every $i\in\mathbb{N}$ there holds

$$\begin{aligned}\int_{T_E(Q)}|\Theta_i 1|^2\delta_E^{2\upsilon-(m-d)}\,d\mu &\leq c_i\,(1-\eta)\,\sigma(\widetilde{Q})+c_i\sigma(Q\setminus\widetilde{Q})+C\sigma(Q)\\ &\leq c_i\,\sigma(Q)-c_i\eta\,\sigma(\widetilde{Q})+C\sigma(Q)\\ &\leq c_i\,\sigma(Q)-c_i\eta C_1^{-1}\,\sigma(Q)+C\sigma(Q)\\ &= c_i\,(1-\eta C_1^{-1})\,\sigma(Q)+C\sigma(Q),\quad \forall\,Q\in\mathbb{D}(E), \end{aligned}\tag{3.107}$$

where for the last inequality in (3.107) we have used (3.88). Dividing both sides of (3.107) by $\sigma(Q)$ and then taking the supremum over all $Q\in\mathbb{D}(E)$ we obtain $c_i\leq c_i\,(1-\eta C_1^{-1})+C$ and furthermore, since $c_i\leq Ci^{4\upsilon}<\infty$, that $c_i\leq\eta^{-1}C_1C$ for all $i\in\mathbb{N}$. This finishes the proof of (3.100).

Having established this, for each $Q\in\mathbb{D}(E)$ we may then write, using (3.94) and Lebesgue's Monotone Convergence Theorem,

$$\begin{aligned}\int_{T_E(Q)}|(\Theta 1)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x) &= \lim_{i\to\infty}\int_{T_E(Q)}|(\Theta_i 1)(x)|^2\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\\ &\leq (\sup_{i\in\mathbb{N}}c_i)\,\sigma(Q)\leq C\sigma(Q),\end{aligned}\tag{3.108}$$

for some finite constant $C>0$ independent of Q. This completes the proof of (3.5) and finishes the proof of Theorem 3.7. $\square$

4 An Inductive Scheme for Square Function Estimates

We now apply the local $T(b)$ Theorem from the previous section to establish an inductive scheme for square function estimates. More specifically, we show that an integral operator Θ, associated with an Ahlfors-David regular set E as in (3.4), satisfies square function estimates whenever the set E contains (uniformly, at all scales and locations) so-called big pieces of sets on which square function estimates for Θ hold. In short, we say that big pieces of square function estimates (BPSFE) imply square function estimates (SFE). We emphasize that this "big pieces" functor is applied to square function estimates for an individual, fixed Θ. Thus, the result to be proved in this section is not a consequence of the stability of UR sets under the big pieces functor, as our particular square function bounds may *not* be equivalent to the property that E is UR.

We continue to work in the context introduced at the beginning of Section 3, except we must assume in addition that the integral kernel θ is not adapted to a fixed set E. In particular, fix two real numbers d,m such that $0<d<m$, and an m-dimensional ADR space $(\mathscr{X},\rho,\mu)$. In this context, suppose that

$$\begin{gathered}\theta:(\mathscr{X}\times\mathscr{X})\setminus\{(x,x):\,x\in\mathscr{X}\}\longrightarrow\mathbb{R}\\ \text{is Borel measurable with respect to the product topology }\tau_\rho\times\tau_\rho,\end{gathered}\tag{4.1}$$

and has the property that there exist finite positive constants C_θ, α, υ such that for all $x,y\in\mathscr{X}$

with $x \neq y$ the following hold:

$$|\theta(x,y)| \leq \frac{C_\theta}{\rho(x,y)^{d+\upsilon}}, \tag{4.2}$$

$$|\theta(x,y) - \theta(x,\widetilde{y})| \leq C_\theta \frac{\rho(y,\widetilde{y})^\alpha}{\rho(x,y)^{d+\upsilon+\alpha}}, \quad \forall\, \widetilde{y} \in \mathscr{X} \setminus \{x\} \text{ with } \rho(y,\widetilde{y}) \leq \tfrac{1}{2}\rho(x,y). \tag{4.3}$$

Then for each closed subset E of $(\mathscr{X}, \tau_\rho)$, and each Borel regular measure σ on $(E, \tau_{\rho|_E})$ with the property that $(E, \rho\big|_E, \sigma)$ is a d-dimensional ADR space, define the integral operator Θ_E for all functions $f \in L^p(E,\sigma)$, $1 \leq p \leq \infty$, by

$$(\Theta_E f)(x) := \int_E \theta(x,y) f(y)\, d\sigma(y), \qquad \forall\, x \in \mathscr{X} \setminus E. \tag{4.4}$$

We begin by defining what it means for a set to have big pieces of square function estimates.

Definition 4.1. *Consider two numbers $d, m \in (0,\infty)$ such that $m > d$, suppose that $(\mathscr{X}, \rho, \mu)$ is an m-dimensional* ADR *space, and assume that θ is as in (4.1)-(4.3). In this context, a set $E \subseteq \mathscr{X}$ is said to have* `Big Pieces of Square Function Estimate` *(or, simply* `BPSFE`*) relative to the kernel θ provided the following conditions are satisfied:*

(i) the set E is closed in $(\mathscr{X}, \tau_\rho)$ and has the property that there exists a Borel regular measure σ on $(E, \tau_{\rho|_E})$ such that $\big(E, \rho\big|_E, \sigma\big)$ is a d-dimensional ADR *space;*

(ii) there exist finite positive constants η, C_1, and C_2 with the property that for each $x \in E$ and each real number $r \in (0, \mathrm{diam}_{\rho_\#}(E)]$ there exists a closed subset $E_{x,r}$ of $(\mathscr{X}, \tau_\rho)$ such that if

$$\sigma_{x,r} := \mathscr{H}^d_{\mathscr{X},\rho_\#} \lfloor E_{x,r}, \qquad \text{where } \mathscr{H}^d_{\mathscr{X},\rho_\#} \text{ is as in (2.29)}, \tag{4.5}$$

then $\big(E_{x,r}, \rho\big|_{E_{x,r}}, \sigma_{x,r}\big)$ is a d-dimensional ADR *space, with* ADR *constant $\leq C_1$, and which satisfies*

$$\sigma\big(E_{x,r} \cap E \cap B_{\rho_\#}(x,r)\big) \geq \eta\, r^d, \tag{4.6}$$

as well as

$$\int_{\mathscr{X} \setminus E_{x,r}} |\Theta_{E_{x,r}} f(z)|^2\, \mathrm{dist}_{\rho_\#}(z, E_{x,r})^{2\upsilon-(m-d)}\, d\mu(z) \leq C_2 \int_{E_{x,r}} |f|^2\, d\sigma_{x,r} \quad \text{for each function } f \in L^2(E_{x,r}, \sigma_{x,r}), \tag{4.7}$$

where $\Theta_{E_{x,r}}$ is the operator associated with $E_{x,r}$ as in (4.4).

In the context of the above definition, the constants η, C_1, C_2 will collectively be referred to as the `BPSFE character` of the set E.

The property of having BPSFE may be dyadically discretized as explained in the lemma below.

Lemma 4.2. *Let $d, m \in (0,\infty)$ be such that $m > d$, assume that $(\mathscr{X}, \rho, \mu)$ is an m-dimensional* ADR *space, and suppose that θ is as in (4.1)-(4.3). In addition, let $E \subseteq \mathscr{X}$ be such that (i) in Definition 4.1 holds and consider the dyadic cube structure $\mathbb{D}(E)$ on E as in Proposition 2.12.*

Then the set E has BPSFE *(relative to θ) if and only if there exist finite positive constants η, C_1, C_2 with the property that for each $Q \in \mathbb{D}(E)$ there exists a closed set $E_Q \subseteq \mathscr{X}$ such that if*

$$\sigma_Q := \mathscr{H}^d_{\mathscr{X},\rho_\#} \lfloor E_Q, \tag{4.8}$$

then $\left(E_Q, \rho\big|_{E_Q}, \sigma_Q\right)$ is a d-dimensional ADR *space with* ADR *constant $\leq C_1$ which satisfies*

$$\mathscr{H}^d_{\mathscr{X},\rho_\#}(E_Q \cap Q) \geq \eta\, \mathscr{H}^d_{\mathscr{X},\rho_\#}(Q) \tag{4.9}$$

as well as

$$\begin{gathered}\int_{\mathscr{X}\setminus E_Q} |\Theta_{E_Q} f(x)|^2 \operatorname{dist}_{\rho_\#}(x, E_Q)^{2\upsilon-(m-d)}\, d\mu(x) \leq C_2 \int_{E_Q} |f|^2\, d\sigma_Q, \\ \textit{for each function } \ f \in L^2(E_Q, \sigma_Q).\end{gathered} \tag{4.10}$$

Proof. The left-to-right implication is a simple consequence of (2.50), while the opposite one follows with the help of (2.53). $\square$

We now state and prove the main result in this section.

Theorem 4.3. *Consider two numbers $d, m \in (0, \infty)$ such that $m > d$, suppose that $(\mathscr{X}, \rho, \mu)$ is an m-dimensional* ADR *space, and assume that θ is as in* (4.1)-(4.3).

If the set $E \subseteq \mathscr{X}$ has BPSFE *relative to θ then there exists a finite constant $C > 0$, depending only on ρ, m, d, υ, C_θ (from* (4.2)-(4.3)*), the* BPSFE *character of E, and the* ADR *constants of E and $\mathscr{X}$, such that*

$$\int_{\mathscr{X}\setminus E} |\Theta_E f(x)|^2\, \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \leq C \int_E |f|^2\, d\sigma, \qquad \forall\, f \in L^2(E, \sigma), \tag{4.11}$$

where

$$\sigma := \mathscr{H}^d_{\mathscr{X},\rho_\#} \lfloor E, \qquad \textit{with } \mathscr{H}^d_{\mathscr{X},\rho_\#} \textit{ as in (2.29).} \tag{4.12}$$

Proof. Since E has BPSFE relative to θ, by Lemma 4.2, for each $Q \in \mathbb{D}(E)$ there exists E_Q satisfying (4.9)-(4.10). For each $Q \in \mathbb{D}(E)$, we then define the function $b_Q : E \to \mathbb{R}$ by setting

$$b_Q(y) := \mathbf{1}_{Q \cap E_Q}(y), \qquad \forall\, y \in E. \tag{4.13}$$

The strategy for proving (4.11) is to invoke Theorem 3.7 for the family $\{b_Q\}_{Q\in\mathbb{D}(E)}$. As such, matters are reduced to checking that conditions *1.–3.* in Theorem 3.7 hold for the collection $\{b_Q\}_{Q\in\mathbb{D}(E)}$ defined in (4.13). Now, condition *1.* is immediate, while the validity of condition *2.* (with $\widetilde{Q} := Q$) is a consequence of (4.9). Thus, it remains to check that condition *3.* holds as well. To this end, fix $Q \in \mathbb{D}(E)$ and for some constant $C_1 \in (1, \infty)$ to be specified later

write (employing notation introduced in (2.19) in relation to both E and E_Q)

$$
\begin{aligned}
&\int_{T_E(Q)} |\Theta_E b_Q(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\\
&\qquad = \int_{T_E(Q)} |\Theta_E b_Q(x)|^2 \mathbf{1}_{\{z\in\mathscr{X}:\,\delta_{E_Q}(z)>C_1\delta_E(z)\}}(x)\,\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\\
&\qquad\quad + \int_{T_E(Q)} |\Theta_E b_Q(x)|^2 \mathbf{1}_{\{z\in\mathscr{X}:\,C_1^{-1}\delta_E(z)\leq\delta_{E_Q}(z)\leq C_1\delta_E(z)\}}(x)\,\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\\
&\qquad\quad + \int_{T_E(Q)} |\Theta_E b_Q(x)|^2 \mathbf{1}_{\{z\in\mathscr{X}:\,\delta_{E_Q}(z)<C_1^{-1}\delta_E(z)\}}(x)\,\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\\
&\qquad =: I_1 + I_2 + I_3.
\end{aligned}
\tag{4.14}
$$

To proceed with estimating I_1 we first obtain a pointwise bound for $\Theta_E b_Q$. To this end, first observe that

$$
\mathcal{O} := \big\{z\in\mathscr{X} : \delta_{E_Q}(z) > C_1\delta_E(z)\big\} \Longrightarrow \mathcal{O}\cap E_Q = \emptyset. \tag{4.15}
$$

Hence, (4.13), (4.2) and (3.19) in Lemma 3.5 give that, for some finite $C>0$ independent of the dyadic cube Q,

$$
\begin{aligned}
|\Theta_E b_Q(x)| &= \left|\int_E \theta(x,y)\,b_Q(y)\,d\sigma(y)\right|\\
&\leq \int_{E_Q} |\theta(x,y)|\,d\sigma(y) \leq \frac{C}{\delta_{E_Q}(x)^{\upsilon}}, \qquad \forall\, x\in\mathcal{O}.
\end{aligned}
\tag{4.16}
$$

Also, (4.9) guarantees that $Q\cap E_Q \neq \emptyset$, and we fix a point $x_0\in Q\cap E_Q$. By (2.145), there exists $c\in(0,\infty)$ such that $T_E(Q)\subseteq B_{\rho_\#}(x_0, c\,\ell(Q))$ which when combined with (4.16) gives

$$
I_1 \leq C\int_{B_{\rho_\#}(x_0,c\ell(Q))\cap\mathcal{O}} \delta_{E_Q}(x)^{-2\upsilon}\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x). \tag{4.17}
$$

At this stage, select a constant $M\in(C_\rho^2,\infty]$, choose $C_1\in(M,\infty)$, and observe that if $x\in\mathcal{O}$ then $\frac{\delta_{E_Q}(x)}{M} > \delta_E(x)$, hence $B_{\rho_\#}\big(x,\delta_{E_Q}(x)/M\big)\cap E\neq\emptyset$. Recalling Lemma 2.11, it follows that there exists $C\in(0,\infty)$ such that

$$
\frac{\delta_{E_Q}(x)^d}{M^d} \leq C\mathscr{H}^d_{\mathscr{X},\rho_\#}\Big(B_{\rho_\#}\Big(x, C_\rho\frac{\delta_{E_Q}(x)}{M}\Big)\cap E\Big), \qquad \forall\, x\in\mathcal{O}. \tag{4.18}
$$

Using this in (4.17) we obtain

$$
I_1 \leq C\int\limits_{B_{\rho_\#}(x_0,c\ell(Q))\cap\mathcal{O}} \Bigg(\int\limits_{B_{\rho_\#}\big(x,C_\rho\delta_{E_Q}(x)/M\big)\cap E} 1\,d\mathscr{H}^d_{\mathscr{X},\rho_\#}(z)\Bigg)\delta_{E_Q}(x)^{-2\upsilon-d}\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x). \tag{4.19}
$$

We make the claim that for each $\vartheta\in(0,1)$

$$
\left.\begin{array}{r}\text{if } x\in\mathscr{X}\setminus E_Q \text{ and } z\in\mathscr{X} \text{ are}\\ \text{such that } \rho_\#(x,z) < \frac{\vartheta}{C_\rho}\delta_{E_Q}(x)\end{array}\right\} \Longrightarrow \frac{1-\vartheta}{C_\rho}\delta_{E_Q}(x) \leq \delta_{E_Q}(z) \leq C_\rho\delta_{E_Q}(x). \tag{4.20}
$$

Indeed, for each $\eta > 1$ close to 1 we may take $y \in E_Q$ satisfying $\rho_{\#}(y,x) < \eta\delta_{E_Q}(x)$ which implies that $\delta_{E_Q}(z) \leq \rho_{\#}(y,z) \leq C_\rho \max\{\rho_{\#}(y,x), \rho_{\#}(x,z)\} \leq C_\rho\eta\delta_{E_Q}(x)$. Upon letting $\eta \searrow 1$ we therefore obtain $\delta_{E_Q}(z) \leq C_\rho\delta_{E_Q}(x)$. On the other hand, for each $w \in E_Q$ we have $\delta_{E_Q}(x) \leq \rho_{\#}(x,w) \leq C_\rho\rho_{\#}(x,z) + C_\rho\rho_{\#}(z,w) \leq \vartheta\delta_{E_Q}(x) + C_\rho\rho_{\#}(z,w)$, which further yields $\delta_{E_Q}(x) \leq \frac{C_\rho}{1-\vartheta}\delta_{E_Q}(z)$. This concludes the proof of (4.20).

Going further, fix $x \in B_{\rho_{\#}}(x_0, c\ell(Q)) \cap \mathcal{O}$ and $z \in B_{\rho_{\#}}\big(x, C_\rho\delta_{E_Q}(x)/M\big) \cap E$ and make two observations. First, an application of (4.20) with $\vartheta := C_\rho^2/M \in (0,1)$ yields

$$\frac{M - C_\rho^2}{MC_\rho}\delta_{E_Q}(x) \leq \delta_{E_Q}(z) \leq C_\rho\delta_{E_Q}(x) \ \text{ and } \ \rho_{\#}(z,x) < \frac{C_\rho}{M}\delta_{E_Q}(x) \leq \frac{C_\rho^2}{M - C_\rho^2}\delta_{E_Q}(z), \tag{4.21}$$

hence $x \in B_{\rho_{\#}}\big(z, \frac{C_\rho^2}{M-C_\rho^2}\delta_{E_Q}(z)\big)$. Second, recalling that $x_0 \in E_Q$, $M > C_\rho^2$ and $C_\rho \geq 1$, we obtain

$$\begin{aligned} \rho_{\#}(x_0,z) &\leq C_\rho \max\big\{\rho_{\#}(x_0,x), \rho_{\#}(x,z)\big\} < C_\rho \max\big\{c\,\ell(Q), \tfrac{C_\rho}{M}\delta_{E_Q}(x)\big\} \\ &\leq C_\rho \max\big\{c\,\ell(Q), \tfrac{1}{C_\rho}\delta_{E_Q}(x)\big\} \leq C_\rho \max\big\{c\,\ell(Q), \tfrac{1}{C_\rho}\rho_{\#}(x_0,x)\big\} = C_\rho c\,\ell(Q), \end{aligned} \tag{4.22}$$

which shows that $z \in B_{\rho_{\#}}(x_0, C_\rho c\ell(Q))$. Combining these observations with (4.21), keeping in mind (4.15), and using Fubini's Theorem in (4.19), we may write

$$\begin{aligned} I_1 &\leq C \int\limits_{B_{\rho_{\#}}\left(x_0, C_\rho c\ell(Q)\right) \cap (E \setminus E_Q)} \delta_{E_Q}(z)^{-2\upsilon-d}\Big(\int\limits_{B_{\rho_{\#}}\left(z, \frac{C_\rho^2}{M-C_\rho^2}\delta_{E_Q}(z)\right)\setminus E_Q} \delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x)\Big)\,d\mathscr{H}^d_{\mathscr{X},\rho_{\#}}(z) \\ &\leq C \int\limits_{B_{\rho_{\#}}\left(x_0, C_\rho c\ell(Q)\right) \cap (E \setminus E_Q)} \delta_{E_Q}(z)^{-2\upsilon-d}\delta_{E_Q}(z)^{2\upsilon+d}\,d\mathscr{H}^d_{\mathscr{X},\rho_{\#}}(z) \\ &\leq C\mathscr{H}^d_{\mathscr{X},\rho_{\#}}\Big(B_{\rho_{\#}}\big(x_0, C_\rho c\,\ell(Q)\big) \cap E\Big) \leq C\ell(Q)^d \leq C\sigma(Q), \end{aligned} \tag{4.23}$$

where for the second inequality in (4.23) we used Lemma 3.6 (with $\gamma := (m-d) - 2\upsilon$ and $r := R := \frac{C_\rho^2}{M-C_\rho^2}\delta_{E_Q}(z)$), and for the last two inequalities the fact that $\big(E, \rho\big|_E, \mathscr{H}^d_{\mathscr{X},\rho_{\#}}\lfloor E\big)$ is d-ADR and $x_0 \in E$.

To estimate I_3, we first note that since $T_E(Q) \cap E = \emptyset$ then (4.2) and (3.19) give that

$$|\Theta_E b_Q(x)| = \left|\int_E \theta(x,y)\,b_Q(y)\,d\sigma(y)\right| \leq \int_E |\theta(x,y)|\,d\sigma(y) \leq \frac{C}{\delta_E(x)^\upsilon}, \quad \forall\, x \in T_E(Q), \tag{4.24}$$

for some finite $C > 0$ independent of Q. Also (compare with (4.16)), $|\Theta_E b_Q(x)| \leq C\delta_{E_Q}(x)^{-\upsilon}$ for each $x \in T_E(Q) \setminus E_Q$. Fix $\alpha, \beta > 0$ such that $\alpha + \beta = \upsilon$. A logarithmically convex combination of these inequalities then yields

$$|\Theta_E b_Q(x)| \leq C\delta_{E_Q}(x)^{-\alpha}\delta_E(x)^{-\beta} \qquad \forall\, x \in T_E(Q) \setminus E_Q. \tag{4.25}$$

Observe that, by assumptions and Lemma 2.10, we have $\mu(E_Q) = 0$. Using this and (4.25) in place of (4.16), we obtain (compare with (4.17))

$$I_3 \leq C\int_{B_{\rho_{\#}}(x_0,c\ell(Q)) \cap (\widetilde{\mathcal{O}} \setminus E_Q)} \delta_E(x)^{-2\beta+2\upsilon-(m-d)}\delta_{E_Q}(x)^{-2\alpha}\,d\mu(x), \tag{4.26}$$

where, this time, we have set

$$\widetilde{\mathcal{O}} := \big\{z \in \mathscr{X} : \delta_E(z) > C_1 \delta_{E_Q}(z)\big\}. \tag{4.27}$$

Given the nature of (4.26), (4.27), the same reasoning leading up to (4.23) used with E and E_Q interchanged this time gives

$$\begin{aligned} I_3 &\leq C \int\limits_{B_{\rho_\#}\left(x_0, C_\rho c\ell(Q)\right)\cap(E_Q\setminus E)} \delta_E(z)^{-2\beta+2\upsilon-m}\Big(\int\limits_{B_{\rho_\#}\left(z,\frac{C_\rho^2}{M-C_\rho^2}\delta_E(z)\right)\setminus E} \delta_E(x)^{-2\alpha}\,d\mu(x)\Big)\,d\mathscr{H}^d_{\mathscr{X},\rho_\#}(z) \\ &\leq C \int\limits_{B_{\rho_\#}\left(x_0, C_\rho c\ell(Q)\right)\cap(E_Q\setminus E)} \delta_E(z)^{-2\beta+2\upsilon-m}\delta_E(z)^{-2\alpha+m}\,d\mathscr{H}^d_{\mathscr{X},\rho_\#}(z) \\ &\leq C\mathscr{H}^d_{\mathscr{X},\rho_\#}\Big(B_{\rho_\#}\big(x_0, C_\rho c\,\ell(Q)\big)\cap E_Q\Big) \leq C\ell(Q)^d \leq C\sigma(Q). \end{aligned} \tag{4.28}$$

Above, the second inequality follows from Lemma 3.6 (used with $r := R := \frac{C_\rho^2}{M-C_\rho^2}\delta_E(z)$) provided we choose $0 < \alpha < (m-d)/2$ to begin with. Also, for the last two inequalities in (4.28) we have made use of the fact that both $\big(E, \rho\big|_E, \mathscr{H}^d_{\mathscr{X},\rho_\#}\lfloor E\big)$ and $\big(E_Q, \rho\big|_{E_Q}, \mathscr{H}^d_{\mathscr{X},\rho_\#}\lfloor E_Q\big)$ are d-ADR spaces and that $x_0 \in E \cap E_Q$.

We are left with estimating I_2. With C_1 as above, thanks to (4.13), (4.4), and (2.32) we have

$$I_2 = \int_{T_E(Q)\setminus E_Q} |\Theta_{E_Q} b_Q(x)|^2 \mathbf{1}_{\{z\in\mathscr{X}:\,C_1^{-1}\delta_E(z)\leq\delta_{E_Q}(z)\leq C_1\delta_E(z)\}}(x)\,\delta_E(x)^{2\upsilon-(m-d)}\,d\mu(x). \tag{4.29}$$

Hence, we may further use (4.29) and (4.10) in order to write (with σ_Q as in (4.8))

$$\begin{aligned} I_2 &\leq C\int_{\mathscr{X}\setminus E_Q} |\Theta_{E_Q} b_Q(x)|^2\,\delta_{E_Q}(x)^{2\upsilon-(m-d)}\,d\mu(x) \\ &\leq C\int_{E_Q} |b_Q|^2\,d\sigma_Q \leq C\mathscr{H}^d_{\mathscr{X},\rho_\#}\big(Q\cap E_Q\big) \leq C\sigma(Q), \end{aligned} \tag{4.30}$$

which is of the correct order.

Now the fact that condition *3* in Theorem 3.7 is satisfied for our choice of b_Q's follows by combining (4.14), (4.23), (4.28) and (4.30). This finishes the proof of the theorem. $\square$

We conclude this section by taking a closer look at a higher order version of the notion of "big pieces of square function estimates." To set the stage, in the context of Definition 4.1 let us say that a closed subset E of $(\mathscr{X}, \tau_\rho)$ with the property that there exists a Borel regular measure σ on $(E, \tau_{\rho|_E})$ such that $\big(E, \rho\big|_E, \sigma\big)$ is a d-dimensional ADR space has `(BP)`^{0}`SFE relative to` θ, or simply `SFE relative to` θ ("Square Function Estimates relative to θ"), provided there exists a finite positive constant C such that

$$\int_{\mathscr{X}\setminus E} |\Theta_E f(z)|^2\,\mathrm{dist}_{\rho_\#}(z,E)^{2\upsilon-(m-d)}\,d\mu(z) \leq C\int_E |f|^2\,d\sigma \quad \text{for each function } f \in L^2(E,\sigma). \tag{4.31}$$

In addition, we shall say that E has $\texttt{(BP)}^1\texttt{SFE}$ whenever E has $\texttt{BPSFE}$.

We may then iteratively interpret "E has $\texttt{(BP)}^{k+1}\texttt{SFE}$" as the property that E contains big pieces of sets having $\texttt{(BP)}^k\texttt{SFE}$, in a uniform fashion. More specifically, we make the following definition.

Definition 4.4. *Consider two numbers $d, m \in (0, \infty)$ such that $m > d$, suppose that $(\mathscr{X}, \rho, \mu)$ is an m-dimensional* ADR *space, and assume that θ is as in* (4.1)-(4.3)*. Also, suppose that $k \in \mathbb{N}$. In this context, a set $E \subseteq \mathscr{X}$ is said to have $\texttt{(BP)}^{k+1}\texttt{SFE}$ relative to θ provided the following conditions are satisfied:*

(i) the set E is closed in $(\mathscr{X}, \tau_\rho)$ and has the property that there exists a Borel regular measure σ on $(E, \tau_{\rho|_E})$ such that $\big(E, \rho\big|_E, \sigma\big)$ is a d-dimensional ADR *space;*

(ii) there exist finite positive constants η, C_1, and C_2 with the property that for each $x \in E$ and each real number $r \in (0, \operatorname{diam}_{\rho_\#}(E)]$ there exists a closed subset $E_{x,r}$ of $(\mathscr{X}, \tau_\rho)$ such that if

$$\sigma_{x,r} := \mathscr{H}^d_{\mathscr{X}, \rho_\#} \lfloor E_{x,r}, \qquad \text{where } \mathscr{H}^d_{\mathscr{X}, \rho_\#} \text{ is as in (2.29)}, \tag{4.32}$$

then $\big(E_{x,r}, \rho\big|_{E_{x,r}}, \sigma_{x,r}\big)$ is a d-dimensional ADR *space, with* ADR *constant $\leq C_1$, and which satisfies*

$$\sigma\big(E_{x,r} \cap E \cap B_{\rho_\#}(x, r)\big) \geq \eta\, r^d, \tag{4.33}$$

as well as

$$E_{x,r} \text{ has } \texttt{(BP)}^k\texttt{SFE} \text{ relative to } \theta, \text{ with character controlled by } C_2. \tag{4.34}$$

In this context, we shall refer to η, C_1, C_2 as the $\texttt{(BP)}^{k+1}\texttt{SFE}$ character of E.

The following result may be regarded as a refinement of Theorem 4.3.

Theorem 4.5. *Consider two numbers $d, m \in (0, \infty)$ such that $m > d$, suppose that $(\mathscr{X}, \rho, \mu)$ is an m-dimensional* ADR *space, and assume that θ is as in* (4.1)-(4.3)*. Also, suppose that the set E is closed in $(\mathscr{X}, \tau_\rho)$ and has the property that there exists a Borel regular measure σ on $(E, \tau_{\rho|_E})$ such that $\big(E, \rho\big|_E, \sigma\big)$ is a d-dimensional* ADR *space.*

Then the following claims are equivalent:

(i) E has $\texttt{(BP)}^k\texttt{SFE}$ relative to θ for some $k \in \mathbb{N}$;

(ii) E has $\texttt{(BP)}^k\texttt{SFE}$ relative to θ for every $k \in \mathbb{N}$;

(iii) E has $\texttt{(BP)}^0\texttt{SFE}$ relative to θ.

Proof. It is clear that if E has $\texttt{(BP)}^k\texttt{SFE}$ relative to θ for some $k \in \mathbb{N}_0$ then E also has $\texttt{(BP)}^{k+1}\texttt{SFE}$ relative to θ, since obviously E has big pieces of itself. This gives the implications *(iii)* $\Rightarrow$ *(i)* and *(iii)* $\Rightarrow$ *(ii)* in the statement of the theorem. Also, the implication *(ii)* $\Rightarrow$ *(i)* is trivial. Finally, in light of Definition 4.4, Theorem 4.3 combined with an induction argument gives that *(i)* $\Rightarrow$ *(iii)*, completing the proof. $\square$

5 Square Function Estimates on Uniformly Rectifiable Sets

Given an n-dimensional Ahlfors-David regular set Σ in $\mathbb{R}^{n+1}$ that has so-called big pieces of Lipschitz graphs (BPLG), the inductive scheme established in the previous section allows us to deduce square function estimates for an integral operator Θ_Σ, as in (4.4), whenever square function estimates are satisfied by Θ_Γ for all Lipschitz graphs Γ in $\mathbb{R}^{n+1}$. Furthermore, induction allows us to prove the same result when the set Σ only has $(\mathrm{BP})^k\mathrm{LG}$ for any $k \in \mathbb{N}$. The definition of $(\mathrm{BP})^k\mathrm{LG}$ is given in Definition 5.5. A recent result by J. Azzam and R. Schul (cf. [8, Corollary 1.7]) proves that uniformly rectifiable sets have $(\mathrm{BP})^2\mathrm{LG}$ (the converse implication also holds and can be found in [26, p. 16]), and this allows us to obtain square function estimates on uniformly rectifiable sets.

We work in the Euclidean codimension one setting throughout this section. In particular, fix $n \in \mathbb{N}$ and let $\mathbb{R}^{n+1}$ be the ambient space, so that in the notation of Section 4, we would have $d = n$, $m = n+1$ and $(\mathscr{X}, \rho, \mu)$ is $\mathbb{R}^{n+1}$ with the Euclidean metric and Lebesgue measure. We also restrict our attention to the following class of kernels in order to obtain square function estimates on Lipschitz graphs. Suppose that $K : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ satisfies

$$K \in C^2(\mathbb{R}^{n+1} \setminus \{0\}), \quad K(\lambda x) = \lambda^{-n} K(x) \text{ for all } \lambda > 0,\ x \in \mathbb{R}^{n+1} \setminus \{0\}, \quad K \text{ is odd}, \tag{5.1}$$

and has the property that there exists a finite positive constant C_K such that for all $j \in \{0, 1, 2\}$ the following holds:

$$|\nabla^j K(x)| \le C_K |x|^{-n-j}, \quad \forall\, x \in \mathbb{R}^{n+1} \setminus \{0\}. \tag{5.2}$$

Then for each closed subset Σ of $\mathbb{R}^{n+1}$, denote by $\sigma := \mathscr{H}^n_{\mathbb{R}^{n+1}} \lfloor \Sigma$ the surface measure induced by the n-dimensional Hausdorff measure on Σ from (2.29), and define the integral operator $\mathcal{T}$ for all functions $f \in L^p(\Sigma, \sigma)$, $1 \le p \le \infty$, by

$$\mathcal{T} f(x) := \int_\Sigma K(x-y) f(y)\, d\sigma(y), \qquad \forall\, x \in \mathbb{R}^{n+1} \setminus \Sigma. \tag{5.3}$$

In the notation of Section 4, we consider the set $E = \Sigma$ and the operator $\Theta_E = \nabla \mathcal{T}$ with integral kernel $\theta = \nabla K$. We begin by proving square function estimates for $\nabla \mathcal{T}$ in the case when Σ is a Lipschitz graph. The inductive scheme from the previous section then allows us to extend that result to the case when Σ has $(\mathrm{BP})^k\mathrm{LG}$ for any $k \in \mathbb{N}$, and hence when Σ is uniformly rectifiable.

5.1 Square function estimates on Lipschitz graphs

The main result in this subsection is the square function estimate for Lipschitz graphs contained in the theorem below. A parabolic variant of this result appears in [42], and the present proof is based on the arguments given there, and in [37].

Theorem 5.1. *Let $A : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function and set $\Sigma := \{(x, A(x)) : x \in \mathbb{R}^n\}$. Moreover, assume that K is as in* (5.1) *and consider the operator $\mathcal{T}$ as in* (5.3)*. Then there exists a finite constant $C > 0$ depending only on $\|\partial^\alpha K\|_{L^\infty(S^n)}$ for $|\alpha| \le 2$, and the Lipschitz constant of A such that for each function $f \in L^2(\Sigma, \sigma)$ one has*

$$\int_{\mathbb{R}^{n+1} \setminus \Sigma} |(\nabla \mathcal{T} f)(x)|^2 \operatorname{dist}(x, \Sigma)\, dx \le C \int_\Sigma |f|^2\, d\sigma. \tag{5.4}$$

As a preamble to the proof of Theorem 5.1, we state and prove a couple of technical lemmas. The first has essentially appeared previously in [13], and is based upon ideas of [54].

Lemma 5.2. *Assume that $A:\mathbb{R}^n\to\mathbb{R}$ is a locally integrable function such that $\nabla A\in L^2(\mathbb{R}^n)$. Pick a smooth, real-valued, nonnegative, compactly supported function ϕ defined in $\mathbb{R}^n$ with $\int_{\mathbb{R}^n}\phi(x)\,dx=1$ and for each $t>0$ set $\phi_t(x):=t^{-n}\phi(x/t)$ for $x\in\mathbb{R}^n$. Finally, define*

$$E_A(t,x,y):=A(x)-A(y)-\langle\nabla_x(\phi_t*A)(x),(x-y)\rangle,\qquad\forall\,x,y\in\mathbb{R}^n,\ \forall\,t>0.\tag{5.5}$$

Then, for some finite positive constant $C=C(\phi,n)$,

$$\int_0^\infty t^{-n-2}\int_{\mathbb{R}^n}\int_{|x-y|\le\lambda t}|E_A(t,x,y)|^2\,dy\,dx\,\tfrac{dt}{t}\le C\lambda^{n+3}\|\nabla A\|^2_{L^2(\mathbb{R}^n)},\qquad\forall\,\lambda\ge1.\tag{5.6}$$

Proof. Starting with the changes of variables $t=\lambda^{-1}\tau$, $y=x+h$ and then employing Plancherel's theorem in the variable x, we may write (with 'hat' denoting the Fourier transform)

$$\begin{aligned}
&\int_0^\infty t^{-n-2}\int_{\mathbb{R}^n}\int_{|x-y|\le\lambda t}|E_A(t,x,y)|^2\,dy\,dx\,\frac{dt}{t}\qquad(5.7)\\
&\quad=\lambda^{n+2}\int_0^\infty\tau^{-n-2}\int_{\mathbb{R}^n}\int_{|h|\le\tau}\big|A(x)-A(x+h)+\langle\nabla_x(\phi_{\lambda^{-1}\tau}*A)(x),h\rangle\big|^2\,dh\,dx\,\frac{d\tau}{\tau}\\
&\quad=\lambda^{n+2}\int_0^\infty\tau^{-n-2}\int_{\mathbb{R}^n}\int_{|h|\le\tau}\big|1-e^{i\langle\zeta,h\rangle}+i\langle\zeta,h\rangle\,\widehat{\phi}(\lambda^{-1}\tau\zeta)\big|^2\tfrac{|\widehat{\nabla A}(\zeta)|^2}{|\zeta|^2}\,dh\,d\zeta\,\frac{d\tau}{\tau}\\
&\quad=\lambda^{n+2}\int_0^\infty\int_{\mathbb{R}^n}\int_{|w|\le1}\frac{\big|1-e^{i\tau\langle\zeta,w\rangle}+i\tau\langle\zeta,w\rangle\,\widehat{\phi}(\lambda^{-1}\tau\zeta)\big|^2}{\tau^2|\zeta|^2}|\widehat{\nabla A}(\zeta)|^2\,dw\,d\zeta\,\frac{d\tau}{\tau},
\end{aligned}$$

where the last equality in (5.7) is based on the change of variables $h=\tau w$.

Next we observe that for every $\zeta\in\mathbb{R}^n$ and $w\in\mathbb{R}^n$ with $|w|\le1$ there holds

$$\frac{\big|1-e^{i\tau\langle\zeta,w\rangle}+i\tau\langle\zeta,w\rangle\,\widehat{\phi}(\lambda^{-1}\tau\zeta)\big|}{\tau|\zeta|}\le C\,\min\Big\{\tau|\zeta|,\frac{\lambda}{\tau|\zeta|}\Big\}\tag{5.8}$$

for some $C>0$ depending only on ϕ. To see why (5.8) is true, analyze the following two cases.

Case 1. $\tau|\zeta|\le\sqrt{\lambda}$:
In this situation the minimum in the right hand-side of (5.8) is equal to $\tau|\zeta|$. In addition, if we use Taylor expansions about zero for the complex exponential function and $\widehat{\phi}$, we obtain (keeping in mind that $\widehat{\phi}(0)=1$, $\lambda\ge1$ and $|w|\le1$)

$$\big|1-e^{i\tau\langle\zeta,w\rangle}+i\tau\langle\zeta,w\rangle+i\tau\langle\zeta,w\rangle\,(\widehat{\phi}(\lambda^{-1}\tau\zeta)-1)\big|\le C\tau^2|\zeta|^2,\tag{5.9}$$

which shows that (5.8) holds in this case.

Case 2. $\tau|\zeta|>\sqrt{\lambda}$:
In this scenario the minimum in the right hand-side of (5.8) is equal to $\frac{\lambda}{\tau|\zeta|}$. Moreover,

$$\begin{aligned}
\big|1-e^{i\tau\langle\zeta,w\rangle}+\tau\langle\zeta,w\rangle\,\widehat{\phi}(\lambda^{-1}\tau\zeta)\big|&\le 2+\tau|\zeta|\,\big|\widehat{\phi}(\lambda^{-1}\tau\zeta)\big|\\
&\le 2+C\tau|\zeta|\,(1+\lambda^{-1}\tau|\zeta|)^{-1}\le C\lambda,
\end{aligned}\tag{5.10}$$

since the Schwartz function $\widehat{\phi}$ decays, $\lambda \geq 1$ and $|w| \leq 1$. Consequently, (5.8) holds in this case as well.

With (5.8) in hand, we proceed to integrate in $\tau \in (0, \infty)$ with respect to the Haar measure to further obtain

$$\int_0^\infty \frac{\left|1 - e^{i\tau\langle\zeta,w\rangle} + i\tau\langle\zeta,w\rangle\,\widehat{\phi}(\lambda^{-1}\tau\zeta)\right|^2}{\tau^2|\zeta|^2}\,\frac{d\tau}{\tau} \leq \int_0^\infty \min\Big\{\tau^2|\zeta|^2, \frac{\lambda^2}{\tau^2|\zeta|^2}\Big\}\,\frac{d\tau}{\tau}$$

$$= \int_0^{\sqrt{\lambda}/|\zeta|} \tau|\zeta|^2\,d\tau + \int_{\sqrt{\lambda}/|\zeta|}^\infty \frac{\lambda^2}{\tau^3|\zeta|^2}\,d\tau \leq C\lambda. \tag{5.11}$$

A combination of (5.7), (5.11) and Plancherel's theorem now yields (5.6), finishing the proof of Lemma 5.2. □

The second lemma needed here has essentially appeared previously in [63].

Lemma 5.3. *Let $F : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ be a continuous function which is even and positive homogeneous of degree $-n-1$. Then for any $a \in \mathbb{R}^n$ and any $t > 0$ there holds*

$$\begin{aligned}\int_{\mathbb{R}^n} F(y, a\cdot y + t)\,dy &= \frac{1}{2t}\int_{S^{n-1}}\int_{-\infty}^\infty F(\omega, s)\,ds\,d\omega \\ &= \int_{\mathbb{R}^n} F(y,t)\,dy.\end{aligned} \tag{5.12}$$

In particular, if F is some first-order partial derivative, say $F = \partial_j G$, $j \in \{1, .., n+1\}$, of a function $G \in C^1(\mathbb{R}^{n+1} \setminus \{0\})$ which is odd and homogeneous of degree $-n$, then

$$\int_{\mathbb{R}^n} F(y, a\cdot y + t)\,dy = 0 \qquad \text{for any } a \in \mathbb{R}^n \text{ and } t > 0. \tag{5.13}$$

Proof. Fix $a \in \mathbb{R}^n$ and $t > 0$. By the homogeneity of F we have $|F(x)| \leq \|F\|_{L^\infty(S^n)}|x|^{-n-1}$ for every $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Also, $\{(y, a\cdot y + t) : |y| \leq 1\}$ is a compact subset of $\mathbb{R}^{n+1} \setminus \{0\}$. Hence, given that F is continuous on $\mathbb{R}^{n+1} \setminus \{0\}$, it follows that $\int_{\mathbb{R}^n} |F(y, a\cdot y + t)|\,dy < \infty$.

To proceed, by passing to polar coordinates $y = r\omega$, $r > 0$, $\omega \in S^{n-1}$, and using the homogeneity of F we may write

$$\int_{\mathbb{R}^n} F(y, a\cdot y + t)\,dy = \int_{S^{n-1}}\int_0^\infty F(\omega, a\cdot\omega + t/r)\,r^{-2}dr\,d\omega. \tag{5.14}$$

Now, setting $s := a\cdot\omega + t/r$ the last integral above becomes $t^{-1}\int_{S^{n-1}}\int_{a\cdot\omega}^\infty F(\omega, s)\,ds\,d\omega$ which, by making the change of variables $(\omega, s) \mapsto (-\omega, -s)$ and using the fact that F is even, may be written as

$$\frac{1}{t}\int_{S^{n-1}}\int_{a\cdot\omega}^\infty F(\omega, s)\,ds\,d\omega = \frac{1}{2t}\int_{S^{n-1}}\int_{-\infty}^\infty F(\omega, s)\,ds\,d\omega. \tag{5.15}$$

This analysis gives the first equality in (5.12). Furthermore, the integral in the right side of (5.15) is independent of $a \in \mathbb{R}^n$ and, hence, so is the original one. In particular, its value does not change if we take $a = 0$ and this is precisely what the second equality in (5.12) says.

Finally, consider the claim made in (5.13) under the assumption that $F = \partial_j G$ for some $j \in \{1, .., n+1\}$ and some $G \in C^1(\mathbb{R}^{n+1} \setminus \{0\})$ which is odd and homogeneous of degree $-n$. In particular, there exists $C \in (0, \infty)$ such that $|G(x)| \leq C|x|^{-n}$ for all $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Then, using the decay of G and integration by parts, if $j \neq n+1$ the third integral in (5.12) vanishes whereas if $j = n+1$ the second one does so. □

After this preamble, we are ready to present the

Proof of Theorem 5.1. A moment's reflection shows that it suffices to establish (5.4) with the domain of integration $\mathbb{R}^{n+1}\setminus\Sigma$ in the left-hand side replaced by

$$\Omega := \{(x,t)\in\mathbb{R}^{n+1} : t > A(x)\}. \tag{5.16}$$

Assume that this is the case and note that by making the bi-Lipschitz change of variables $\mathbb{R}^n\times(0,\infty)\ni(x,t)\mapsto(x,A(x)+t)\in\Omega$, (whose Jacobian is equivalent to a finite constant) the estimate (5.4) follows from the boundedness of

$$T^j : L^2(\mathbb{R}^n,dx)\to L^2(\mathbb{R}^{n+1}_+,\tfrac{dt}{t}dx), \tag{5.17}$$

$$T^j f(x,t) := \int_{\mathbb{R}^n} K^j_t(x,y)f(y)\,dy \tag{5.18}$$

for $j=1,..,n+1$, where the family of kernels $\{K^j_t(x,y)\}_{t>0}$ is given by

$$K^j_t(x,y) := t\,(\partial_j K)(x-y,A(x)-A(y)+t), \qquad x,y\in\mathbb{R}^n,\ t>0,\ j=1,\dots,n+1. \tag{5.19}$$

The approach we present utilizes ideas developed in [15] and [37]. Based on (5.1)-(5.2) it is not difficult to check that the family $\{K^j_t(x,y)\}_{t>0}$ is standard, i.e., there hold

$$|K^j_t(x,y)| \ \leq\ C\,t(t+|x-y|)^{-(n+1)} \tag{5.20}$$

$$|\nabla_x K^j_t(x,y)|+|\nabla_y K^j_t(x,y)| \ \leq\ C\,t(t+|x-y|)^{-(n+2)}. \tag{5.21}$$

As such, a particular version of Theorem 3.2 gives that the operators in (5.17) are bounded as soon as we show that for each $j=1,\dots,n+1$,

$$|T^j(1)(x,t)|^2\,\tfrac{dt}{t}dx \text{ is a Carleson measure in } \mathbb{R}^{n+1}_+. \tag{5.22}$$

To this end, fix $j\in\{1,\dots,n+1\}$ and select a real-valued, nonnegative function $\phi\in C^\infty_c(\mathbb{R}^n)$, vanishing for $|x|\geq 1$, with $\int_{\mathbb{R}^n}\phi(x)\,dx=1$ and, as usual, for every $t>0$, set $\phi_t(x):=t^{-n}\phi(x/t)$ for $x\in\mathbb{R}^n$. We write $T^j(1)=(T^j(1)-\widetilde{T}^j(1))+\widetilde{T}^j(1)$ where

$$\widetilde{T}^j f(x,t) := \int_{\mathbb{R}^n}\widetilde{K}^j_t(x,y)f(y)\,dy, \qquad x\in\mathbb{R}^n,\ t>0, \tag{5.23}$$

with

$$\widetilde{K}^j_t(x,y) := t\,(\partial_j K)(x-y,\langle\nabla_x(\phi_t * A)(x),x-y\rangle+t), \qquad x,y\in\mathbb{R}^n,\ t>0. \tag{5.24}$$

To prove that $|(T^j-\widetilde{T}^j)(1)(x,t)|^2\,dx\frac{dt}{t}$ is a Carleson measure, fix x_0 in $\mathbb{R}^n$, $r>0$, and split

$$(T^j-\widetilde{T}^j)(1) = (T^j-\widetilde{T}^j)(\mathbf{1}_{B(x_0,100r)}) + (T^j-\widetilde{T}^j)(\mathbf{1}_{\mathbb{R}^n\setminus B(x_0,100r)}), \tag{5.25}$$

where, for any set S, $\mathbf{1}_S$ stands for the characteristic function of S. Using (5.20) and the fact that a similar estimate holds for $\widetilde{K}^j_t(x,y)$, we may write

$$\begin{aligned}
&\int_0^r\int_{B(x_0,r)}|(T^j-\widetilde{T}^j)(\mathbf{1}_{\mathbb{R}^n\setminus B(x_0,100r)})(x,t)|^2\,dx\,\tfrac{dt}{t}\\
&\qquad\leq C\int_0^r\int_{B(x_0,r)}\Big(\int_{\mathbb{R}^n\setminus B(x_0,100r)}\frac{t}{|x-y|^{n+1}}\,dy\Big)^2\,dx\,\tfrac{dt}{t}\\
&\qquad= C\int_0^r\int_{B(x_0,r)}\Big(\int_{\mathbb{R}^n\setminus B(0,99r)}\frac{t}{|z|^{n+1}}\,dz\Big)^2\,dx\,\tfrac{dt}{t} = C\,r^n,
\end{aligned} \tag{5.26}$$

a bound which is of the right order. We are therefore left with proving an estimate similar to (5.26) with $\mathbb{R}^n \setminus B(x_0, 100r)$ replaced by $B(x_0, 100r)$. More precisely, the goal is to show that

$$\int_0^r \int_{B(x_0,r)} |(T^j - \widetilde{T}^j)(\mathbf{1}_{B(x_0,100r)})(x,t)|^2 \, dx \, \tfrac{dt}{t} \leq C \, r^n. \tag{5.27}$$

For this task we make use of the Lemma 5.2. This requires an adjustment which we now explain. Concretely, fix a function $\Phi \in C^\infty(\mathbb{R})$ such that $0 \leq \Phi \leq 1$, $\operatorname{supp} \Phi \subseteq [-150r, 150r]$, $\Phi \equiv 1$ on $[-125r, 125r]$, and $\|\Phi'\|_{L^\infty(\mathbb{R})} \leq c/r$. If we now set $\widetilde{A}(x) := \Phi(|x - x_0|)(A(x) - A(x_0))$ for every $x \in \mathbb{R}^n$, it follows that

$$\begin{gathered} \widetilde{A}(x) - \widetilde{A}(y) = A(x) - A(y) \quad \text{and} \quad \nabla(\phi_t * \widetilde{A})(x) = \nabla(\phi_t * A)(x) \\ \text{whenever} \quad x \in B(x_0, r), \ y \in B(x_0, 100r), \ t \in (0, r). \end{gathered} \tag{5.28}$$

Hence, the expression $(T^j - \widetilde{T}^j)(\mathbf{1}_{B(x_0,100r)})(x,t)$ does not change for $x \in B(x_0, r)$ and $t \in (0, r)$ if we replace A by $\widetilde{A}$. In addition, since $\|\nabla \widetilde{A}\|_{L^\infty(\mathbb{R}^n)} \leq C \|\nabla A\|_{L^\infty(\mathbb{R}^n)}$, taking into account the support of $\widetilde{A}$ we have

$$\|\nabla \widetilde{A}\|_{L^2(\mathbb{R}^n)} \leq C r^{n/2} \|\nabla A\|_{L^\infty(\mathbb{R}^n)} \tag{5.29}$$

for some $C > 0$ independent of r. Hence, there is no loss of generality in assuming that the original Lipschitz function A has the additional property that

$$\|\nabla A\|_{L^2(\mathbb{R}^n)} \leq C r^{n/2} \|\nabla A\|_{L^\infty(\mathbb{R}^n)}. \tag{5.30}$$

Under this assumption we now return to the task of proving (5.27). To get started, recall (5.5). We claim that there exists $C = C(A, \phi) > 0$ such that

$$|K_t^j(x,y) - \widetilde{K}_t^j(x,y)| \leq C \, t(t + |x-y|)^{-(n+2)} |E_A(t,x,y)|, \quad \forall \, x, y \in \mathbb{R}^n, \, \forall \, t > 0. \tag{5.31}$$

Indeed, by making use of the Mean-Value Theorem and (5.2), the claim will follow if we show that there exists $C = C(A, \phi) > 0$ with the property that

$$\sup_{\xi \in I} [|\xi| + |x-y|]^{-(n+2)} \leq C[t + |x-y|]^{-(n+2)}, \tag{5.32}$$

where I denotes the interval with endpoints $t + A(x) - A(y)$ and $t + \langle \nabla_x(\phi_t * A)(x), (x-y) \rangle$. From the properties of A and ϕ we see that $\xi = t + O(|x-y|)$, with constants depending only on A and ϕ. In particular, there exists some small $\varepsilon = \varepsilon(A, \phi) > 0$ such that if $|x-y| < \varepsilon t$ then $t \leq C|\xi| \leq C(|\xi| + |x-y|)$. On the other hand, if $|x-y| \geq \varepsilon t$ then clearly $t \leq C(|\xi| + |x-y|)$. Thus, there exists $C > 0$ such that $t \leq C(|\xi| + |x-y|)$ for $\xi \in I$, which implies that for some $C = C(A, \phi) > 0$ there holds $t + |x-y| \leq C(|\xi| + |x-y|)$ whenever $\xi \in I$, proving (5.32).

Next, making use of (5.31), we may write

$$\int_0^r \int_{B(x_0,r)} |(T^j - \widetilde{T}^j)(\mathbf{1}_{B(x_0,100r)})(x,t)|^2\, dx\, \frac{dt}{t}$$

$$\leq C \int_0^\infty \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \frac{t}{(t+|x-y|)^{n+2}}\, |E_A(t,x,y)|\, dy \Big)^2 dx\, \frac{dt}{t}$$

$$\leq C \int_0^\infty \int_{\mathbb{R}^n} \Big(t^{-n-1} \int_{B(x,t)} |E_A(t,x,y)|\, dy \Big)^2 dx\, \frac{dt}{t}$$

$$+ C \int_0^\infty \int_{\mathbb{R}^n} \Big(\sum_{\ell=0}^\infty \int\limits_{B(x,2^{\ell+1}t)\setminus B(x,2^\ell t)} \frac{t}{|x-y|^{n+2}}\, |E_A(t,x,y)|\, dy \Big)^2 dx\, \frac{dt}{t}$$

$$\leq C \int_0^\infty \int_{\mathbb{R}^n} \Big(\sum_{\ell=0}^\infty 2^{-\ell} (2^\ell t)^{-n-1} \int\limits_{B(x,2^{\ell+1}t)} |E_A(t,x,y)|\, dy \Big)^2 dx\, \frac{dt}{t}. \tag{5.33}$$

Now, we apply Minkowski's inequality in order to obtain

$$\int_0^\infty \int_{\mathbb{R}^n} \Big(\sum_{\ell=0}^\infty 2^{-\ell} (2^\ell t)^{-n-1} \int\limits_{B(x,2^{\ell+1}t)} |E_A(t,x,y)|\, dy \Big)^2 dx\, \frac{dt}{t}$$

$$\leq \left(\sum_{\ell=0}^\infty \Big[\int_0^\infty \int_{\mathbb{R}^n} 2^{-2\ell} (2^\ell t)^{-2n-2} \Big(\int\limits_{B(x,2^{\ell+1}t)} |E_A(t,x,y)|\, dy \Big)^2 dx\, \frac{dt}{t} \Big]^{1/2} \right)^2. \tag{5.34}$$

By the Cauchy-Schwarz inequality, the last expression above is dominated by

$$\left(\sum_{\ell=0}^\infty \Big[2^{\ell(-n-4)} \int_0^\infty \int_{\mathbb{R}^n} t^{-n-2} \int\limits_{B(x,2^{\ell+1}t)} |E_A(t,x,y)|^2\, dy\, dx\, \frac{dt}{t} \Big]^{1/2} \right)^2. \tag{5.35}$$

Invoking now Lemma 5.2 with $\lambda := 2^{\ell+1} \geq 1$ for $\ell \in \mathbb{N} \cup \{0\}$, each inner triple integral in (5.35) is dominated by $C 2^{\ell(n+3)} \|\nabla A\|^2_{L^2(\mathbb{R}^n)}$ with $C > 0$ finite constant independent of ℓ. Thus, the entire expression in (5.35) is

$$\leq C \left(\sum_{\ell=0}^\infty \big[2^{\ell(-n-4)} \cdot 2^{\ell(n+3)} \|\nabla A\|^2_{L^2(\mathbb{R}^n)} \big]^{1/2} \right)^2 = C \|\nabla A\|^2_{L^2(\mathbb{R}^n)} \leq C r^n, \tag{5.36}$$

where for the last inequality in (5.36) we have used (5.30). This finishes the proof of (5.27). In turn, when (5.27) is combined with (5.26), we obtain

$$|(T^j - \widetilde{T}^j)(1)(x,t)|^2\, \tfrac{dt}{t} dx \text{ is a Carleson measure in } \mathbb{R}^{n+1}_+. \tag{5.37}$$

At this stage, there remains to observe that, thanks to Lemma 5.3, we have

$$\widetilde{T}^j(1)(x,t) = \int\limits_{\mathbb{R}^n} t\, (\partial_j K)\big(x-y, \langle \nabla_x(\phi_t * A)(x), (x-y)\rangle + t\big)\, dy \equiv 0 \quad \forall\, x \in \mathbb{R}^n,\ \forall\, t > 0. \tag{5.38}$$

The proof of Theorem 5.1 is now completed. $\square$

5.2 Square function estimates on $(\mathrm{BP})^k$LG sets

We continue to work in the context of $\mathbb{R}^{n+1}$ introduced at the beginning of Section 5, and abbreviate the n-dimensional Hausdorff outer measure from Definition 2.3 as $\mathcal{H}^n := \mathcal{H}^n_{\mathbb{R}^{n+1}}$. We prove that square function estimates are stable under the so-called big pieces functor. Square function estimates on uniformly rectifiable sets then follow as a simple corollary. Let us begin by reviewing the concept of uniform rectifiability. In particular, following G. David and S. Semmes [25], we make the following definition.

Definition 5.4. *A closed set $\Sigma \subseteq \mathbb{R}^{n+1}$ is called* `uniformly rectifiable` *provided it is n-dimensional Ahlfors-David regular and the following property holds. There exist ε, $M \in (0,\infty)$ (called the* UR *constants of Σ) such that for each $x \in \Sigma$ and $r > 0$, there is a Lipschitz map $\varphi : B^n_r \to \mathbb{R}^{n+1}$ (where B^n_r is a ball of radius r in $\mathbb{R}^n$) with Lipschitz constant at most equal to M, such that*

$$\mathcal{H}^n\big(\Sigma \cap B(x,r) \cap \varphi(B^n_r)\big) \geq \varepsilon r^n. \tag{5.39}$$

If Σ is compact, then this is only required for $r \in (0, \operatorname{diam}(\Sigma)]$.

There are a variety of equivalent characterizations of uniform rectifiability (cf., e.g., [26, Theorem I.1.5.7, p. 22]); the version above is often specified by saying that Σ has `Big Pieces of Lipschitz Images` (or, simply `BPLI`). Another version, in which Lipschitz maps are replaced with Bi-Lipschitz maps, is specified by saying that Σ has `Big Pieces of Bi-Lipschitz Images` (or, simply `BPBI`). The equivalence between BPLI and BPBI can be found in [26, p. 22]. We also require the following notion of sets having big pieces of Lipschitz graphs.

Definition 5.5. *A set $\Sigma \subseteq \mathbb{R}^{n+1}$ is said to have* `Big Pieces of Lipschitz Graphs` *(or, simply* `BPLG`*) provided it is n-dimensional Ahlfors-David regular and the following property holds. There exist ε, $M \in (0,\infty)$ (called the* BPLG *constants of Σ) such that for each $x \in \Sigma$ and $r > 0$, there is an n-dimensional Lipschitz graph $\Gamma \subseteq \mathbb{R}^{n+1}$ with Lipschitz constant at most equal to M, such that*

$$\mathcal{H}^n\big(\Sigma \cap B(x,r) \cap \Gamma\big) \geq \varepsilon r^n. \tag{5.40}$$

If Σ is compact, then this is only required for $r \in (0, \operatorname{diam}(\Sigma)]$.

We also write `(BP)`^{1}`LG` *to mean* `BPLG`*. For each $k \geq 1$, a set $\Sigma \subseteq \mathbb{R}^{n+1}$ is said to have* `Big Pieces of (BP)`^{k}`LG` *(or, simply* `(BP)`$^{k+1}$`LG`*) provided it is n-dimensional Ahlfors-David regular and the following property holds. There exist δ, ε, $M \in (0,\infty)$ (called the* $(\mathrm{BP})^{k+1}$LG *constants of Σ) such that for each $x \in \Sigma$ and $r > 0$, there is a set $\Omega \subseteq \mathbb{R}^{n+1}$ that has* $(\mathrm{BP})^k$LG *with* ADR *constant at most equal to M, and* $(\mathrm{BP})^k$LG *constants ε, M, such that*

$$\mathcal{H}^n\big(\Sigma \cap B(x,r) \cap \Omega\big) \geq \delta r^n. \tag{5.41}$$

If Σ is compact, then this is only required for $r \in (0, \operatorname{diam}(\Sigma)]$.

We now combine the inductive scheme from Section 4 with the square function estimates for Lipschitz graphs from Subsection 5.1 to prove that square function estimates are stable under the so-called big pieces functor.

Theorem 5.6. *Let $k \in \mathbb{N}$ and suppose that $\Sigma \subseteq \mathbb{R}^{n+1}$ has (BP)^{k}LG. Let K be a real-valued kernel satisfying* (5.1)*, and let $\mathcal{T}$ denote the integral operator associated with Σ as in* (5.3)*. Then there exists a constant $C \in (0,\infty)$ depending only on n, the (BP)^{k}LG constants of Σ, and $\|\partial^\alpha K\|_{L^\infty(S^n)}$ for $|\alpha| \leq 2$, such that*

$$\int_{\mathbb{R}^{n+1}\setminus\Sigma} |\nabla \mathcal{T} f(x)|^2 \operatorname{dist}(x,\Sigma)\, dx \leq C \int_\Sigma |f|^2\, d\sigma, \qquad \forall\, f \in L^2(\Sigma,\sigma), \tag{5.42}$$

where $\sigma := \mathscr{H}^n \lfloor \Sigma$ is the measure induced by the n-dimensional Hausdorff measure on Σ.

Proof. The proof proceeds by induction on $\mathbb{N}$. For the case $k = 1$, suppose that $\Sigma \subseteq \mathbb{R}^{n+1}$ has BPLG with BPLG constants ε_0, $C_0 \in (0,\infty)$. For each $x \in \Sigma$ and $r > 0$, there is an n-dimensional Lipschitz graph $\Gamma \subseteq \mathbb{R}^{n+1}$ with Lipschitz constant at most equal to C_0, such that

$$\mathcal{H}^n\big(\Sigma \cap B(x,r) \cap \Gamma\big) \geq \varepsilon_0 r^n. \tag{5.43}$$

It follows from Theorem 5.1 that Σ has BPSFE with BPSFE character (cf. Definition 4.1) depending only on the BPLG constants of Σ, and $\|\partial^\alpha K\|_{L^\infty(S^n)}$ for $|\alpha| \leq 2$. It then follows from Theorem 4.3 that (5.42) holds for some $C \in (0,\infty)$ depending only on n, the BPLG constants of Σ, and $\|\partial^\alpha K\|_{L^\infty(S^n)}$ for $|\alpha| \leq 2$.

Now let $j \in \mathbb{N}$ and assume that the statement of the theorem holds in the case $k = j$. Suppose that $\Sigma \subseteq \mathbb{R}^{n+1}$ has (BP)$^{j+1}$LG with (BP)$^{j+1}$LG constants ε_1, ε_2, $C_1 \in (0,\infty)$. For each $x \in \Sigma$ and $r > 0$, there is a set $\Omega \subseteq \mathbb{R}^{n+1}$ that has (BP)^{j}LG with ADR constant at most equal to C_1, and (BP)^{j}LG constants ε_1, C_1, such that

$$\mathcal{H}^n\big(\Sigma \cap B(x,r) \cap \Omega\big) \geq \varepsilon_2 r^n. \tag{5.44}$$

It follows by the inductive assumption that Σ has BPSFE with BPSFE character depending only on the constants specified in the theorem in the case $k = j$. Applying again Theorem 4.3 we obtain that (5.42) holds for some $C \in (0,\infty)$ depending only on n, the (BP)$^{j+1}$LG constants of Σ, and $\|\partial^\alpha K\|_{L^\infty(S^n)}$ for $|\alpha| \leq 2$. This completes the proof. ☐

The recent result by J. Azzam and R. Schul (cf. [8, Corollary 1.7]) that uniformly rectifiable sets have (BP)^{2}LG allows us to obtain the following as an immediate corollary of Theorem 5.6.

Corollary 5.7. *Suppose that $\Sigma \subseteq \mathbb{R}^{n+1}$ is a uniformly rectifiable set. Let K be a real-valued kernel satisfying* (5.1)*, and let $\mathcal{T}$ denote the integral operator associated with Σ as in* (5.3)*. Then there exists a constant $C \in (0,\infty)$, depending only on n, the UR constants of Σ, and $\|\partial^\alpha K\|_{L^\infty(S^n)}$ for $|\alpha| \leq 2$, such that*

$$\int_{\mathbb{R}^{n+1}\setminus\Sigma} |\nabla \mathcal{T} f(x)|^2 \operatorname{dist}(x,\Sigma)\, dx \leq C \int_\Sigma |f|^2\, d\sigma, \qquad \forall\, f \in L^2(\Sigma,\sigma), \tag{5.45}$$

where $\sigma := \mathscr{H}^n \lfloor \Sigma$ is the measure induced by the n-dimensional Hausdorff measure on Σ.

Proof. The set Σ has (BP)^{2}LG by J. Azzam and R. Schul's characterization of uniformly rectifiable sets in [8, Corollary 1.7], so the result follows at once from Theorem 5.6. ☐

5.3 Square function estimates for integral operators with variable kernels

The square function estimates from Theorem 5.6 and Corollary 5.7 have been formulated for *convolution type* integral operators and our goal in this subsection is to prove some versions of these results which apply to integral operators with variable coefficient kernels. A first result in this regard reads as follows.

Theorem 5.8. *Let $k \in \mathbb{N}$ and suppose that $\Sigma \subseteq \mathbb{R}^{n+1}$ is compact and has (BP)^{k}LG. Then there exists a positive integer $M = M(n)$ with the following significance. Assume that $\mathcal{U}$ is a bounded, open neighborhood of Σ in $\mathbb{R}^{n+1}$ and consider a function*

$$\mathcal{U} \times \big(\mathbb{R}^{n+1} \setminus \{0\}\big) \ni (x,z) \mapsto b(x,z) \in \mathbb{R} \tag{5.46}$$

which is odd and (positively) homogeneous of degree $-n$ in the variable $z \in \mathbb{R}^{n+1} \setminus \{0\}$, and which has the property that

$$\partial_x^\beta \partial_z^\alpha b(x,z) \text{ is continuous and bounded on } \mathcal{U} \times S^n \text{ for } |\alpha| \leq M \text{ and } |\beta| \leq 1. \tag{5.47}$$

Finally, define the variable kernel integral operator

$$\mathcal{B}f(x) := \int_\Sigma b(x, x-y) f(y)\, d\sigma(y), \qquad x \in \mathcal{U} \setminus \Sigma, \tag{5.48}$$

where $\sigma := \mathscr{H}^n \lfloor \Sigma$ is the measure induced by the n-dimensional Hausdorff measure on Σ.

Then there exists a constant $C \in (0,\infty)$ depending only on n, the (BP)^{k}LG constants of Σ, the diameter of $\mathcal{U}$, and $\|\partial_x^\beta \partial_z^\alpha b\|_{L^\infty(\mathcal{U}\times S^n)}$ for $|\alpha| \leq 2$, $|\beta| \leq 1$, such that

$$\int_{\mathcal{U}\setminus\Sigma} |\nabla \mathcal{B} f(x)|^2 \operatorname{dist}(x,\Sigma)\, dx \leq C \int_\Sigma |f|^2\, d\sigma, \qquad \forall\, f \in L^2(\Sigma,\sigma). \tag{5.49}$$

In particular, (5.49) holds whenever Σ is uniformly rectifiable (while retaining the other background assumptions).

In preparation for presenting the proof of Theorem 5.8, we state two lemmas, of geometric character, from [61].

Lemma 5.9. *Let $(\mathscr{X},\rho)$ be a geometrically doubling quasi-metric space and let $\Sigma \subseteq \mathscr{X}$ be a set with the property that $\big(\Sigma, \rho\big|_\Sigma, \mathcal{H}^d_{\mathscr{X},\rho}\lfloor\Sigma\big)$ becomes a d-dimensional* ADR *space, for some $d > 0$. Assume that μ is a Borel measure on $\mathscr{X}$ satisfying*

$$\sup_{x\in\mathscr{X},\, r>0} \frac{\mu\big(B_{\rho_\#}(x,r)\big)}{r^m} < +\infty, \tag{5.50}$$

for some $m \geq 0$. Also, fix a constant $c > 0$ and select two real numbers N, α such that $\alpha < m - d$ and $N < m - \max\{\alpha, 0\}$.

Then there exists a constant $C \in (0,\infty)$ depending on the supremum in (5.50), the geometric doubling constant of $(\mathscr{X},\rho)$, the ADR *constant of Σ, as well as N, α, and c, such that*

$$\int_{B_{\rho_\#}(x,r)\setminus\overline{\Sigma}} \frac{\operatorname{dist}_{\rho_\#}(y,\Sigma)^{-\alpha}}{\rho_\#(x,y)^N}\, d\mu(y) \leq C\, r^{m-\alpha-N}, \tag{5.51}$$

$$\forall\, r > 0, \quad \forall\, x \in \mathscr{X} \quad \text{with} \quad \operatorname{dist}_{\rho_\#}(x,\Sigma) < c\,r.$$

Lemma 5.10. *Let $(\mathscr{X},\rho)$ be a quasi-metric space. Suppose $E\subseteq\mathscr{X}$ is nonempty and σ is a measure on E such that $(E,\rho\big|_E,\sigma)$ becomes a d-dimensional* ADR *space, for some $d>0$. Fix a real number $0\le N<d$. Then there exists $C\in(0,\infty)$ depending only on N, ρ, and the* ADR *constant of E such that*

$$\int\limits_{E\cap B_{\rho_\#}(x,r)}\frac{1}{\rho_\#(x,y)^N}\,d\sigma(y)\le C\,r^{d-N},\qquad\forall\,x\in\mathscr{X},\quad\forall\,r>\operatorname{dist}_{\rho_\#}(x,E).\tag{5.52}$$

We are now ready to discuss the

Proof of Theorem 5.8. Set

$$H_0:=1,\quad H_1:=n+1,\quad\text{ and }\quad H_\ell:=\binom{n+\ell}{\ell}-\binom{n+\ell-2}{\ell-2}\quad\text{ if }\ \ell\ge 2,\tag{5.53}$$

and, for each $\ell\in\mathbb{N}_0$, let $\big\{\Psi_{i\ell}\big\}_{1\le i\le H_\ell}$ be an orthonormal basis for the space of spherical harmonics of degree ℓ on the n-dimensional sphere S^n. In particular,

$$H_\ell\le(\ell+1)\cdot(\ell+2)\cdots(n+\ell-1)\cdot(n+\ell)\le C_n\,\ell^n\quad\text{ for }\ \ell\ge 2\tag{5.54}$$

and, if Δ_{S^n} denotes the Laplace-Beltrami operator on S^n, then for each $\ell\in\mathbb{N}_0$ and $1\le i\le H_\ell$,

$$\Delta_{S^n}\Psi_{i\ell}=-\ell(n+\ell-1)\Psi_{i\ell}\ \text{ on }\ S^n,\ \text{ and }\ \Psi_{i\ell}\Big(\frac{x}{|x|}\Big)=\frac{P_{i\ell}(x)}{|x|^\ell}\tag{5.55}$$

for some homogeneous harmonic polynomial $P_{i\ell}$ of degree ℓ in $\mathbb{R}^{n+1}$. Also,

$$\big\{\Psi_{i\ell}\big\}_{\ell\in\mathbb{N}_0,\,1\le i\le H_\ell}\ \text{ is an orthonormal basis for }\ L^2(S^n),\tag{5.56}$$

hence,

$$\|\Psi_{i\ell}\|_{L^2(S^n)}=1\ \text{ for each }\ell\in\mathbb{N}_0\text{ and }1\le i\le H_\ell.\tag{5.57}$$

More details on these matters may be found in, e.g., [72, pp. 137–152] and [71, pp. 68–75].

Assume next that an even integer $d>(n/2)+2$ has been fixed. Sobolev's embedding theorem then gives that for each $\ell\in\mathbb{N}_0$ and $1\le i\le H_\ell$ (with I standing for the identity on S^n)

$$\|\Psi_{i\ell}\|_{C^2(S^n)}\le C_n\big\|(I-\Delta_{S^n})^{d/2}\Psi_{i\ell}\big\|_{L^2(S^n)}\le C_n\ell^d,\tag{5.58}$$

where the last inequality is a consequence of (5.55)-(5.57).

Fix $\ell\in\mathbb{N}_0$ and $1\le i\le H_\ell$ arbitrary. If we now define

$$a_{i\ell}(x):=\int_{S^n}b(x,\omega)\Psi_{i\ell}(\omega)\,d\omega,\quad\text{for each }\ x\in\mathcal{U},\tag{5.59}$$

it follows from the last formula in (5.55) and the assumptions on $b(x,z)$ that

$$a_{i\ell}\text{ is identically zero whenever }\ell\text{ is even.}\tag{5.60}$$

Also, for each number $N \in \mathbb{N}$ with $2N \leq M$ and each multiindex β of length ≤ 1 we have

$$\begin{aligned}
\sup_{x\in\mathcal{U}}\big|[-\ell(n+\ell-1)]^N(\partial^\beta a_{i\ell})(x)\big| &= \sup_{x\in\mathcal{U}}\Big|\int_{S^n}(\partial_x^\beta b)(x,\omega)\big(\Delta_{S^n}^N\Psi_{i\ell}\big)(\omega)\,d\omega\Big| \\
&= \sup_{x\in\mathcal{U}}\Big|\int_{S^n}\big(\partial_x^\beta\Delta_{S^n}^N b\big)(x,\omega)\Psi_{i\ell}(\omega)\,d\omega\Big| \\
&\leq \sup_{x\in\mathcal{U}}\big\|\big(\partial_x^\beta\Delta_{S^n}^N b\big)(x,\cdot)\big\|_{L^2(S^n)} \\
&\leq C_n \sup_{\substack{(x,z)\in\mathcal{U}\times S^n\\ |\alpha|\leq M}}\big|\big(\partial_x^\beta\partial_z^\alpha b\big)(x,z)\big| =: C_b, \qquad (5.61)
\end{aligned}$$

where C_b is a finite constant. Hence, for each number $N \in \mathbb{N}$ with $2N \leq M$ there exists a constant $C_{n,N}$ such that

$$\sup_{x\in\mathcal{U},\,|\beta|\leq 1}\big|(\partial^\beta a_{i\ell})(x)\big| \leq C_{n,N}C_b\,\ell^{-2N}, \qquad \ell\in\mathbb{N}_0, \quad 1\leq i\leq H_\ell. \tag{5.62}$$

For each fixed $x \in \mathcal{U}$, expand the function $b(x,\cdot) \in L^2(S^n)$ with respect to the orthonormal basis $\{\Psi_{i\ell}\}_{\ell\in\mathbb{N}_0,\,1\leq i\leq H_\ell}$ in order to obtain that (in the sense of $L^2(S^n)$ in the variable $z/|z| \in S^n$)

$$\begin{aligned}
b(x,z) &= b\Big(x,\frac{z}{|z|}\Big)|z|^{-n} = \sum_{\ell\in\mathbb{N}}\sum_{i=1}^{H_\ell}a_{i\ell}(x)\Psi_{i\ell}\Big(\frac{z}{|z|}\Big)|z|^{-n} \\
&= \sum_{\ell\in 2\mathbb{N}+1}\sum_{i=1}^{H_\ell}a_{i\ell}(x)\Psi_{i\ell}\Big(\frac{z}{|z|}\Big)|z|^{-n}, \qquad (5.63)
\end{aligned}$$

where the last equality is a consequence of (5.60). For each $\ell \in 2\mathbb{N}+1$ let us now set

$$k_{i\ell}(z) := \Psi_{i\ell}\Big(\frac{z}{|z|}\Big)|z|^{-n}, \quad z\in\mathbb{R}^{n+1}\setminus\{0\}, \tag{5.64}$$

so that, if d is as in (5.58), then for each $|\alpha| \leq 2$ we have

$$\|\partial^\alpha k_{i\ell}\|_{L^\infty(S^n)} \leq C_n\|\Psi_{i\ell}\|_{C^2(S^n)} \leq C_n\ell^d. \tag{5.65}$$

Also, given any $f \in L^2(\Sigma,\sigma)$, set

$$\mathcal{B}_{i\ell}f(x) := \int_\Sigma k_{i\ell}(x-y)f(y)\,d\sigma(y), \quad x\in\mathcal{U}\setminus\Sigma, \tag{5.66}$$

and note that for any compact subset $\mathcal{O}$ of $\mathcal{U}\setminus\Sigma$ and any multiindex α with $|\alpha| \leq 1$,

$$\sup_{x\in\mathcal{O}}\big|\big(\partial^\alpha\mathcal{B}_{i\ell}f\big)(x)\big| \leq C(n,\mathcal{O},\Sigma)\ell^d, \tag{5.67}$$

by (5.65). On the other hand, if $N > (d+1)/2$ (a condition which we shall assume from now on) then (5.58) and (5.62) imply that the last series in (5.63) converges to $b(x,z)$ uniformly for $x \in \mathcal{U}$ and z in compact subsets of $\mathbb{R}^{n+1}\setminus\{0\}$. As such, it follows from (5.64) and (5.66) that

$$\mathcal{B}f(x) = \sum_{\ell\in 2\mathbb{N}+1}\sum_{i=1}^{H_\ell}a_{i\ell}(x)\mathcal{B}_{i\ell}f(x), \quad \text{uniformly on compact subsets of } \mathcal{U}\setminus\Sigma. \tag{5.68}$$

Using this, (5.67) and (5.62), the term-by-term differentiation theorem for series of functions may be invoked in order to obtain that

$$\big(\nabla\mathcal{B}f\big)(x)=\sum_{\ell\in 2\mathbb{N}+1}\sum_{i=1}^{H_\ell}a_{i\ell}(x)\big(\nabla\mathcal{B}_{i\ell}f\big)(x)+\sum_{\ell\in 2\mathbb{N}+1}\sum_{i=1}^{H_\ell}(\nabla a_{i\ell})(x)\mathcal{B}_{i\ell}f(x),$$
$$\text{uniformly for } x \text{ in compact subsets of } \mathcal{U}\setminus\Sigma. \tag{5.69}$$

Moving on, observe that for each $\ell\in 2\mathbb{N}+1$ and $1\le i\le H_\ell$, Theorem 5.6 gives

$$\int_{\mathcal{U}\setminus\Sigma}|\nabla\mathcal{B}_{i\ell}f(x)|^2\,\mathrm{dist}(x,\Sigma)\,dx\le C_{i\ell}\int_\Sigma|f|^2\,d\sigma,\qquad\forall\, f\in L^2(\Sigma,\sigma), \tag{5.70}$$

where, with $C\in(0,\infty)$ depending only on the dimension n and the (BP)^{k}LG constants of Σ.

$$C_{i\ell}=C\max_{|\alpha|\le 2}\|\partial^\alpha k_{i\ell}\|_{L^\infty(S^n)}\le C\ell^d, \tag{5.71}$$

thanks to (5.58). Thus, if

$$M\in\mathbb{N} \text{ is odd and satisfies } M>d+1, \tag{5.72}$$

one may choose $N\in\mathbb{N}$ such that $d+1<2N<M$. Such a choice ensures that for every $f\in L^2(\Sigma,\sigma)$

$$\begin{aligned}
&\sum_{\ell\in 2\mathbb{N}+1}\sum_{i=1}^{H_\ell}\Big(\int_{\mathcal{U}\setminus\Sigma}|a_{i\ell}(x)|^2|\nabla\mathcal{B}_{i\ell}f(x)|^2\,\mathrm{dist}(x,\Sigma)\,dx\Big)^{1/2}\\
&\qquad\le C_{n,N}C_b\sum_{\ell\in 2\mathbb{N}+1}\sum_{i=1}^{H_\ell}\ell^{-2N}\Big(\int_{\mathcal{U}\setminus\Sigma}|\nabla\mathcal{B}_{i\ell}f(x)|^2\,\mathrm{dist}(x,\Sigma)\,dx\Big)^{1/2}\\
&\qquad\le C_{n,N}C_b\Big(\sum_{\ell\in 2\mathbb{N}+1}\sum_{i=1}^{H_\ell}\ell^{-2N}C_{i\ell}^{1/2}\Big)\Big(\int_\Sigma|f|^2\,d\sigma\Big)^{1/2}\\
&\qquad\le C_{n,N}C_b\Big(\sum_{\ell\in 2\mathbb{N}+1}\sum_{i=1}^{H_\ell}\ell^{d/2-2N}\Big)\Big(\int_\Sigma|f|^2\,d\sigma\Big)^{1/2}\\
&\qquad=C\Big(\int_\Sigma|f|^2\,d\sigma\Big)^{1/2},
\end{aligned}\tag{5.73}$$

by (5.62), (5.71), and our choice of N.

To proceed, let $\ell\in\mathbb{N}$ and $1\le i\le H_\ell$ be arbitrary. Also, fix an arbitrary $f\in L^2(\Sigma,\sigma)$. Then

$$\begin{aligned}
&\Big(\int_{\mathcal{U}\setminus\Sigma}|\nabla a_{i\ell}|^2|\mathcal{B}_{i\ell}f(x)|^2\,\mathrm{dist}(x,\Sigma)\,dx\Big)^{1/2}\\
&\qquad\le C_{n,N}C_b\,\ell^{-2N}\Big(\int_{\mathcal{U}\setminus\Sigma}\big|\mathrm{dist}(x,\Sigma)^{1/2}\mathcal{B}_{i\ell}f(x)\big|^2\,dx\Big)^{1/2}\\
&\qquad=C_{n,N}C_b\,\ell^{-2N}\Big(\int_{\mathcal{U}\setminus\Sigma}\big|\mathcal{T}_{i\ell}f(x)\big|^2\,dx\Big)^{1/2}
\end{aligned}\tag{5.74}$$

where

$$\mathcal{T}_{i\ell} : L^2(\Sigma,\sigma) \longrightarrow L^2(\mathcal{U}\setminus\Sigma) \tag{5.75}$$

is the integral operator whose integral kernel is given by

$$K_{i\ell}(x,y) := \mathrm{dist}(x,\Sigma)^{1/2} k_{i\ell}(x-y), \qquad x\in\mathcal{U}\setminus\Sigma,\ \ y\in\Sigma. \tag{5.76}$$

Note that

$$\begin{aligned} \sup_{x\in\mathcal{U}\setminus\Sigma}\int_\Sigma \big|K_{i\ell}(x,y)\big|\,d\sigma(y) &\leq \|\Psi_{i\ell}\|_{L^\infty(S^n)} \sup_{x\in\mathcal{U}\setminus\Sigma}\int_\Sigma \frac{\mathrm{dist}(x,\Sigma)^{1/2}}{|x-y|^n}\,d\sigma(y) \\ &\leq C\ell^d \sup_{x\in\mathcal{U}\setminus\Sigma}\int_\Sigma \frac{1}{|x-y|^{n-1/2}}\,d\sigma(y) \\ &\leq C\ell^d\,\mathrm{diam}(\mathcal{U})^{1/2}, \end{aligned} \tag{5.77}$$

by (5.58) and Lemma 5.10, and that

$$\begin{aligned} \sup_{y\in\Sigma}\int_{\mathcal{U}\setminus\Sigma} \big|K_{i\ell}(x,y)\big|\,dx &\leq \|\Psi_{i\ell}\|_{L^\infty(S^n)} \sup_{y\in\Sigma}\int_{\mathcal{U}\setminus\Sigma} \frac{\mathrm{dist}(x,\Sigma)^{1/2}}{|x-y|^n}\,dx \\ &\leq C\ell^d\,\mathrm{diam}(\mathcal{U})^{3/2}, \end{aligned} \tag{5.78}$$

by (5.58) and Lemma 5.9. From (5.77)-(5.78) and Schur's Lemma we then deduce that the operator $\mathcal{T}_{i\ell}$ is bounded in the context of (5.75), with norm

$$\big\|\mathcal{T}_{i\ell}\big\|_{L^2(\Sigma,\sigma)\to L^2(\mathcal{U}\setminus\Sigma)} \leq C\ell^d\,\mathrm{diam}(\mathcal{U}). \tag{5.79}$$

Combining (5.79) and (5.74) we therefore arrive at the conclusion that, for each $f\in L^2(\Sigma,\sigma)$,

$$\Big(\int_{\mathcal{U}\setminus\Sigma}|\nabla a_{i\ell}|^2|\mathcal{B}_{i\ell}f(x)|^2\,\mathrm{dist}(x,\Sigma)\,dx\Big)^{1/2} \leq C(\mathcal{U})\ell^d\Big(\int_\Sigma|f|^2\,d\sigma\Big)^{1/2}, \tag{5.80}$$

whenever $\ell\in\mathbb{N}$ and $1\leq i\leq H_\ell$. As a result, there exists $C\in(0,\infty)$ such that

$$\sum_{\ell\in 2\mathbb{N}+1}\sum_{i=1}^{H_\ell}\Big(\int_{\mathcal{U}\setminus\Sigma}|\nabla a_{i\ell}|^2|\mathcal{B}_{i\ell}f(x)|^2\,\mathrm{dist}(x,\Sigma)\,dx\Big)^{1/2} \leq C\Big(\int_\Sigma|f|^2\,d\sigma\Big)^{1/2}, \tag{5.81}$$

for every $f\in L^2(\Sigma,\sigma)$.

Fix now an arbitrary compact subset $\mathcal{O}$ of $\mathcal{U}\setminus\Sigma$. Then (5.69), (5.73) and (5.81) allow us to estimate

$$\Big(\int_{\mathcal{O}}|\nabla\mathcal{B}f(x)|^2\,\mathrm{dist}(x,\Sigma)\,dx\Big)^{1/2} \leq C\Big(\int_\Sigma|f|^2\,d\sigma\Big)^{1/2}, \tag{5.82}$$

where the constant C is independent of $\mathcal{O}$ and $f\in L^2(\Sigma,\sigma)$. Upon letting $\mathcal{O}\nearrow\mathcal{U}\setminus\Sigma$ in (5.82), Lebesgue's Monotone Convergence Theorem then yields (5.49). Finally, the last claim in the statement of Theorem 5.8 is justified in a similar manner, based on Corollary 5.7. $\square$

It is also useful to treat the following variant of (5.48):

$$\widetilde{\mathcal{B}}f(x) := \int\limits_{\Sigma} b(y, x-y) f(y)\, d\sigma(y), \qquad x \in \mathcal{U} \setminus \Sigma. \tag{5.83}$$

The same sort of analysis works, with x replaced by y in the spherical harmonic expansion (5.63) (in fact, the argument is simpler since the $a_{i\ell}$'s act this time as multipliers in the y variable). Specifically, we have the following.

Theorem 5.11. *In the setting of Theorem 5.8, with $\widetilde{\mathcal{B}}$ given by* (5.83) *where, this time, in place of* (5.47) *one assumes*

$$\partial_z^\alpha b(x,z) \text{ is continuous and bounded on } \mathcal{U} \times S^n \text{ for } |\alpha| \leq M, \tag{5.84}$$

there holds

$$\int_{\mathcal{U}\setminus\Sigma} |\nabla \widetilde{\mathcal{B}} f(x)|^2 \operatorname{dist}(x, \Sigma)\, dx \leq C \int_{\Sigma} |f|^2\, d\sigma, \qquad \forall\, f \in L^2(\Sigma, \sigma). \tag{5.85}$$

In turn, Theorem 5.8 and Theorem 5.11 apply to the Schwartz kernels of certain pseudodifferential operators. Recall that a pseudodifferential operator $Q(x, D)$ with symbol $q(x,\xi)$ in Hörmander's class $S^m_{1,0}$ is given by the oscillatory integral

$$\begin{aligned} Q(x,D)u &= (2\pi)^{-(n+1)/2} \int q(x,\xi) \hat{u}(\xi) e^{i\langle x,\, \xi\rangle}\, d\xi \\ &= (2\pi)^{-(n+1)} \iint q(x,\xi) e^{i\langle x-y,\, \xi\rangle} u(y)\, dy\, d\xi. \end{aligned} \tag{5.86}$$

Here, we are concerned with a smaller class of symbols, S^m_{cl}, defined by requiring that the (matrix-valued) function $q(x,\xi)$ has an asymptotic expansion of the form

$$q(x,\xi) \sim q_m(x,\xi) + q_{m-1}(x,\xi) + \cdots, \tag{5.87}$$

with q_j smooth in x and ξ and homogeneous of degree j in ξ (for $|\xi| \geq 1$). Call $q_m(x,\xi)$, i.e. the leading term in (5.87), the *principal symbol* of $q(x,D)$. In fact, we shall find it convenient to work with classes of symbols which only exhibit a limited amount of regularity in the spatial variable (while still C^∞ in the Fourier variable). Specifically, for each $r \geq 0$ we define

$$C^r S^m_{1,0} := \big\{ q(X,\xi) :\, \|D_\xi^\alpha q(\cdot,\xi)\|_{C^r} \leq C_\alpha (1+|\xi|)^{m-|\alpha|}, \quad \forall\, \alpha \big\}. \tag{5.88}$$

Denote by $\mathrm{OP}C^r S^m_{1,0}$ the class of pseudodifferential operators associated with such symbols. As before, we write $\mathrm{OP}C^r S^m_{\mathrm{cl}}$ for the subclass of *classical* pseudodifferential operators in $\mathrm{OP}C^r S^m_{1,0}$ whose symbols can be expanded as in (5.87), where $q_j(x,\xi) \in C^r S^{m-j}_{1,0}$ is homogeneous of degree j in ξ for $|\xi| \geq 1$, $j = m, m-1, \dots$. Finally, we set $\emptyset\mathrm{P}C^r S^m_{\mathrm{cl}}$ for the space of all formal adjoints of operators in $\mathrm{OP}C^r S^m_{\mathrm{cl}}$.

Given a classical pseudodifferential operator $Q(x,D) \in \mathrm{OP}C^r S^{-1}_{\mathrm{cl}}$, we denote by $k_Q(x,y)$ and $\mathrm{Sym}_Q(x,\xi)$ its Schwartz kernel and its principal symbol, respectively. Next, if the sets $\Sigma \subseteq \mathcal{U} \subseteq \mathbb{R}^{n+1}$ are as in Theorem 5.8, we can introduce the integral operator

$$\mathcal{B}_Q f(x) := \int_{\Sigma} k_Q(x,y) f(y)\, d\sigma(y), \qquad x \in \mathcal{U} \setminus \Sigma. \tag{5.89}$$

In this context, Theorem 5.8 and Theorem 5.11 yield the following result.

Theorem 5.12. *Let $\Sigma \subseteq \mathbb{R}^{n+1}$ be compact and uniformly rectifiable, and assume that $\mathcal{U}$ is a bounded, open neighborhood of Σ in $\mathbb{R}^{n+1}$. Let $Q(x,D) \in \mathrm{OPC}^1 S^{-1}_{\mathrm{cl}}$ be such that $\mathrm{Sym}_Q(x,\xi)$ is odd in ξ. Then the operator* (5.89) *satisfies*

$$\int_{\mathcal{U}\setminus\Sigma} |\nabla \mathcal{B}_Q f(x)|^2 \operatorname{dist}(x,\Sigma)\, dx \leq C \int_\Sigma |f|^2\, d\sigma, \qquad \forall\, f \in L^2(\Sigma,\sigma). \tag{5.90}$$

Moreover, a similar result is valid for a pseudodifferential operator $Q(x,D) \in \varnothing\mathrm{PC}^0 S^{-1}_{\mathrm{cl}}$.

In fact, since the main claims in Theorem 5.12 are local in nature and given the invariance of the class of domains and pseudodifferential operators (along with their Schwartz kernels and principal symbols) under smooth diffeomorphisms, these results can be naturally extended to the setting of domains on manifolds and pseudodifferential operators acting between vector bundles. Formulated as such, these in turn extend results proved in [63] for Lipschitz subdomains of Riemannian manifolds.

6 L^p Square Function Estimates

We have so far only considered L^2 square function estimates. We now consider L^p versions for $p \in (0,\infty]$. The natural setting for the consideration of these estimates is in term of mixed norm spaces $L^{(p,q)}(\mathscr{X},E)$, originally introduced in [59] (cf. also [9] for related matters). We begin by using the tools developed in Section 2 to analyze these spaces in the context of an ambient quasi-metric space $\mathscr{X}$ and a closed subset E. In the case $\mathscr{X} = \mathbb{R}^{n+1}$ and $E = \partial\mathbb{R}^{n+1}_+ \approx \mathbb{R}^n$, the mixed norm spaces correspond to the tent spaces introduced by R. Coifman, Y. Meyer and E.M. Stein in [16]. The preliminary analysis in Subsections 6.1 and 6.2 is based on the techniques developed in that paper, although we need to overcome a variety of geometric obstructions that arise outside of the Euclidean setting. We build on this in Subsection 6.3, where we prove that L^2 square function estimates associated with integral operators Θ_E, as defined in Section 3, follow from weak L^p square function estimates for any $p \in (0,\infty)$. This is achieved by combining the $T(1)$ theorem from Subsection 3.1 with a weak type John-Nirenberg lemma for Carleson measures, the Euclidean version of which appears in [4]. The theory culminates in Subsection 6.4, where we prove two extrapolation theorems for estimates associated with integral operators Θ_E, as defined in Section 3. In particular, we prove that a weak L^q square function estimate for any $q \in (0,\infty)$ implies that square functions are bounded from H^p into L^p for all $p \in (\frac{d}{d+\gamma},\infty)$, where H^p is a Hardy space, d is the the dimension of E, and γ is a finite positive constant depending on the ambient space $\mathscr{X}$ and the operator Θ_E.

6.1 Mixed norm spaces

We begin by considering the mixed norm spaces $L^{(p,q)}$ from [59] (cf. also [9]) and then, following the theory of tent spaces in [16], record some extensive preliminaries that are used throughout Section 6. In particular, Theorem 6.8 contains an equivalence for the quasi-norms of the mixed norm spaces that is essential in the next subsection.

Let $(\mathscr{X},\rho)$ be a quasi-metric space, E a nonempty subset of $\mathscr{X}$, and μ a Borel measure on $(\mathscr{X},\tau_\rho)$. Recall the regularized version $\rho_\#$ of the quasi-distance ρ discussed in Theorem 2.2, and recall that we employ the notation $\delta_E(y) = \operatorname{dist}_{\rho_\#}(y,E)$ for each $y \in \mathscr{X}$. Next, let $\kappa > 0$ be arbitrary, fixed, and consider the `nontangential approach regions`

$$\Gamma_\kappa(x) := \big\{ y \in \mathscr{X}\setminus E :\, \rho_\#(x,y) < (1+\kappa)\,\delta_E(y) \big\}, \qquad \forall\, x \in E. \tag{6.1}$$

Occasionally, we shall refer to κ as the `aperture` of the nontangential approach region $\Gamma_\kappa(x)$. Since both $\rho_\#(\cdot,\cdot)$ and $\delta_E(\cdot)$ are continuous (cf. Theorem 2.2) it follows that $\Gamma_\kappa(x)$ is an open subset of $(\mathscr{X},\tau_\rho)$, for each $x\in E$. Furthermore, it may be readily verified that

$$\mathscr{X}\setminus\overline{E}=\bigcup_{x\in E}\Gamma_\kappa(x),\qquad\forall\,\kappa>0,\tag{6.2}$$

where $\overline{E}$ denotes the closure of E in the topology τ_ρ.

Lemma 6.1. *Let $(\mathscr{X},\rho)$ be a quasi-metric space, E a proper, nonempty, closed subset of $(\mathscr{X},\tau_\rho)$, and μ a Borel measure on $(\mathscr{X},\tau_\rho)$. Let $u:\mathscr{X}\setminus E\to[0,\infty]$ be a μ-measurable function, fix $\kappa>0$ and recall the regions from* (6.1)*. Then the function*

$$F:E\longrightarrow[0,\infty],\qquad F(x):=\int_{\Gamma_\kappa(x)}u(y)\,d\mu(y),\quad\forall\,x\in E,\tag{6.3}$$

is lower semi-continuous (relative to the topology induced by τ_ρ on E).

Proof. Let $x_0\in E$ be arbitrary, fixed, and consider a sequence $\{x_j\}_{j\in\mathbb{N}}$ of points in E with the property that

$$\lim_{j\to\infty}\rho_\#(x_j,x_0)=0.\tag{6.4}$$

We claim that

$$\liminf_{j\to\infty}\mathbf{1}_{\Gamma_\kappa(x_j)}(x)\geq\mathbf{1}_{\Gamma_\kappa(x_0)}(x),\qquad\forall\,x\in\mathscr{X}\setminus E.\tag{6.5}$$

Clearly (6.5) is true if $x\notin\Gamma_\kappa(x_0)$. If $\mathbf{1}_{\Gamma_\kappa(x_0)}(x)=1$, then $x\in\Gamma_\kappa(x_0)$, thus by definition $\rho_\#(x,x_0)<(1+\kappa)\delta_E(x)$. Based on the continuity of $\rho_\#(x,\cdot)$ and (6.4), it follows that there exists $j_0\in\mathbb{N}$ such that $\rho_\#(x,x_j)<(1+\kappa)\delta_E(x)$ for $j\geq j_0$. Hence, $x\in\Gamma_\kappa(x_j)$ for $j\geq j_0$ or, equivalently, $\mathbf{1}_{\Gamma_\kappa(x_j)}(x)=1$ for $j\geq j_0$. This completes the proof of the claim.

Returning to the actual task at hand, Fatou's lemma and (6.5) then imply

$$\begin{aligned}\liminf_{j\to\infty}F(x_j)&=\liminf_{j\to\infty}\int_{\mathscr{X}\setminus E}\mathbf{1}_{\Gamma_\kappa(x_j)}u\,d\mu\geq\int_{\mathscr{X}\setminus E}\liminf_{j\to\infty}\big(\mathbf{1}_{\Gamma_\kappa(x_j)}u\big)\,d\mu\\&=\int_{\mathscr{X}\setminus E}\big(\liminf_{j\to\infty}\mathbf{1}_{\Gamma_\kappa(x_j)}\big)u\,d\mu\geq\int_{\mathscr{X}\setminus E}\mathbf{1}_{\Gamma_\kappa(x_0)}u\,d\mu\\&=F(x_0).\end{aligned}\tag{6.6}$$

This shows that F is lower semi-continuous. $\square$

We retain the context of Lemma 6.1. For each index $q\in(0,\infty)$ and constant $\kappa\in(0,\infty)$, define the L^q-based `Lusin operator`, or `area operator`, $\mathscr{A}_{q,\kappa}$ for all μ-measurable functions $u:\mathscr{X}\setminus E\to\overline{\mathbb{R}}:=[-\infty,+\infty]$ by

$$(\mathscr{A}_{q,\kappa}u)(x):=\Big(\int_{\Gamma_\kappa(x)}|u(y)|^q\,d\mu(y)\Big)^{\frac{1}{q}},\qquad\forall\,x\in E.\tag{6.7}$$

As a consequence of Lemma 6.1, we have that $\mathscr{A}_{q,\kappa}u$ is lower semi-continuous, hence

$$\big\{x\in E:\,(\mathscr{A}_{q,\kappa}u)(x)>\lambda\big\}\quad\text{is an open subset of }(E,\tau_\rho)\text{ for each }\lambda>0.\tag{6.8}$$

To proceed, fix a Borel measure σ on $(E, \tau_{\rho|_E})$. The above considerations then allow us to conclude that

$$\begin{gathered}\text{for any } \mu\text{-measurable function } u : \mathscr{X} \setminus E \to \overline{\mathbb{R}}, \\ \text{the mapping } \mathscr{A}_{q,\kappa} u : E \to [0,\infty] \text{ is well-defined and } \sigma\text{-measurable.}\end{gathered} \tag{6.9}$$

Consequently, given $\kappa > 0$ and a pair of integrability indices p, q, following [59] and [9] we may now introduce the `mixed norm space of type` (p,q), denoted $L^{(p,q)}(\mathscr{X}, E, \mu, \sigma; \kappa)$, or $L^{(p,q)}(\mathscr{X}, E)$ for short, in a meaningful manner as follows. If $q \in (0,\infty)$ and $p \in (0,\infty]$ we set

$$L^{(p,q)}(\mathscr{X}, E, \mu, \sigma; \kappa) := \Big\{ u : \mathscr{X} \setminus E \to \overline{\mathbb{R}} : \ u \ \ \mu\text{-measurable and } \ \mathscr{A}_{q,\kappa} u \in L^p(E,\sigma) \Big\}, \tag{6.10}$$

equipped with the quasi-norm

$$\|u\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)} := \|\mathscr{A}_{q,\kappa} u\|_{L^p(E,\sigma)} = \left\{ \begin{array}{ll} \Big(\int_E \Big[\int_{\Gamma_\kappa(x)} |u|^q \, d\mu \Big]^{p/q} d\sigma(x) \Big)^{1/p} & \text{if } \ p < \infty, \\ \sigma\text{-ess}\sup\limits_{x \in E} \, (\mathscr{A}_{q,\kappa} u)(x) \quad \text{if } \ p = \infty. \end{array} \right. \tag{6.11}$$

Also, corresponding to $p \in (0,\infty)$ and $q = \infty$, we set

$$L^{(p,\infty)}(\mathscr{X}, E, \mu, \sigma; \kappa) := \Big\{ u : \mathscr{X} \setminus E \to \overline{\mathbb{R}} : \ \|\mathcal{N}_\kappa u\|_{L^p(E,\sigma)} < \infty \Big\}, \tag{6.12}$$

where $\mathcal{N}_\kappa$ is the nontangential maximal operator defined by

$$(\mathcal{N}_\kappa u)(x) := \sup_{y \in \Gamma_\kappa(x)} |u(y)|, \qquad \forall\, x \in E, \tag{6.13}$$

and equip this space with the quasi-norm $\|u\|_{L^{(p,\infty)}(\mathscr{X},E,\mu,\sigma;\kappa)} := \|\mathcal{N}_\kappa u\|_{L^p(E,\sigma)}$. Finally, corresponding to $p = q = \infty$, set

$$L^{(\infty,\infty)}(\mathscr{X}, E, \mu, \sigma; \kappa) := L^\infty(\mathscr{X} \setminus E, \mu). \tag{6.14}$$

We note that the connection of our mixed norm spaces with the Coifman-Meyer-Stein tent spaces T^p_q in $\mathbb{R}^{n+1}_+$ is as follows

$$T^p_q = L^{(p,q)}\Big(\mathbb{R}^{n+1}, \partial\mathbb{R}^{n+1}_+, \mathbf{1}_{\mathbb{R}^{n+1}_+} \frac{dx\,dt}{t^{n+1}}, dx \Big), \quad \text{for } \ p, q \in (0,\infty). \tag{6.15}$$

Thus, results for mixed normed spaces imply results for classical tent spaces.

The next goal is to clarify to what extent the quasi-norm $\|\cdot\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)}$ depends on the parameter $\kappa > 0$ associated with the nontangential approach regions Γ_k defined in (6.1) and utilized in (6.11), (6.13). This is done in Theorem 6.8 below, but the proof requires a number of preliminary results and definitions which we now present.

To set the stage, for each $A \subseteq E$ and $\kappa > 0$, define the `fan` (or `saw-tooth`) `region` $\mathcal{F}_\kappa(A)$ `above` A, and the `tent region` $\mathcal{T}_\kappa(A)$ `above` A, as

$$\mathcal{F}_\kappa(A) := \bigcup_{x \in A} \Gamma_\kappa(x) \quad \text{ and } \quad \mathcal{T}_\kappa(A) := \big(\mathscr{X} \setminus E \big) \setminus \Big(\mathcal{F}_\kappa(E \setminus A) \Big). \tag{6.16}$$

Also, for each point $y \in \mathscr{X} \setminus E$, define the "(reverse) conical projection" of y onto E by

$$\pi^\kappa_y := \big\{ x \in E : \ y \in \Gamma_\kappa(x) \big\}. \tag{6.17}$$

Lemma 6.2. *Let $(\mathscr{X},\rho)$ be a quasi-metric space, E a proper, nonempty, closed subset of $(\mathscr{X},\tau_\rho)$. For every $A\subseteq E$, denote by $\overline{A}$ and A°, respectively, the closure and interior of A in the topological space $(E,\tau_{\rho|_E})$. Then for each fixed $\kappa\in(0,\infty)$ the following properties hold.*

(i) *For each $A\subseteq E$ one has $\mathcal{F}_\kappa(A)=\mathcal{F}_\kappa(\overline{A})$ and $\mathcal{T}_\kappa(A^\circ)=\mathcal{T}_\kappa(A)$.*

(ii) *For each $A\subseteq E$ one has $\mathcal{T}_\kappa(A)\subseteq\mathcal{F}_\kappa(A)$.*

(iii) *For each nonempty proper subset A of E one has*

$$\mathcal{T}_\kappa(A)=\big\{x\in\mathscr{X}\setminus E:\ \mathrm{dist}_{\rho_\#}(x,A)\leq(1+\kappa)^{-1}\,\mathrm{dist}_{\rho_\#}(x,E\setminus A)\big\},\tag{6.18}$$

$$\mathcal{T}_\kappa(A)=\big\{y\in\mathscr{X}\setminus E:\ \pi_y^\kappa\subseteq A\big\}.\tag{6.19}$$

Moreover, for each nonempty subset A of E one has

$$\mathcal{F}_\kappa(A)=\big\{y\in\mathscr{X}\setminus E:\ \mathrm{dist}_{\rho_\#}(y,A)<(1+\kappa)\,\delta_E(y)\big\}.\tag{6.20}$$

(iv) *One has $\mathcal{F}_\kappa(E)=\mathcal{T}_\kappa(E)=\mathscr{X}\setminus E$. Also, for any family $(A_j)_{j\in J}$ of subsets of E,*

$$\bigcup_{j\in J}\mathcal{F}_\kappa(A_j)=\mathcal{F}_\kappa\big(\cup_{j\in J}A_j\big),\qquad\bigcap_{j\in J}\mathcal{T}_\kappa(A_j)=\mathcal{T}_\kappa\big(\cap_{j\in J}A_j\big),\tag{6.21}$$

and

$$A_1\subseteq A_2\subseteq E\implies\mathcal{F}_\kappa(A_1)\subseteq\mathcal{F}_\kappa(A_2)\quad\textit{and}\quad\mathcal{T}_\kappa(A_1)\subseteq\mathcal{T}_\kappa(A_2).\tag{6.22}$$

(v) *Given $A\subseteq E$, it follows that $\mathcal{F}_\kappa(A)$ is an open subset of $(\mathscr{X},\tau_\rho)$, while $\mathcal{T}_\kappa(A)$ is a relatively closed subset of $\mathscr{X}\setminus E$ equipped with the topology induced by τ_ρ on this set.*

(vi) *For each $y\in\mathscr{X}\setminus E$ it follows that π_y^κ is a relatively open set in the topology induced by τ_ρ on E.*

(vii) *One has*

$$B_{\rho_\#}\big(x,C_\rho^{-1}r\big)\setminus E\subseteq\mathcal{T}_\kappa\big(E\cap B_{\rho_\#}(x,r)\big),\qquad\forall\,r\in(0,\infty),\quad\forall\,x\in E.\tag{6.23}$$

(viii) *Assume that $(E,\rho\big|_E)$ is geometrically doubling. Then for every $\kappa>0$ there exists a constant $C_o\in(0,\infty)$ with the property that if $\mathcal{O}$ is a nonempty, open, proper subset of $(E,\tau_{\rho|_E})$ and if $\{\Delta_j\}_{j\in J}$, where $x_j\in E$ and $\Delta_j:=E\cap B_\rho(x_j,r_j)$ for each $j\in J$, is a Whitney decomposition of $\mathcal{O}$ as in Proposition 2.6, then*

$$\mathcal{T}_\kappa(\mathcal{O})\subseteq\bigcup_{j\in J}B_\rho(x_j,C_or_j).\tag{6.24}$$

In particular, there exists $C\in(0,\infty)$ with the property that

$$\begin{gathered}\mathcal{T}_\kappa\big(E\cap B_\rho(x,r)\big)\subseteq B_\rho(x,Cr)\setminus E\quad\textit{whenever}\\ x\in E\textit{ and }r>0\textit{ are such that }E\setminus B_\rho(x,r)\neq\emptyset.\end{gathered}\tag{6.25}$$

(ix) In the case when E is bounded, there exists $C \in (0,\infty)$ with the property that

$$\mathscr{X} \setminus B_{\rho_\#}\big(x_0, C\,\mathrm{diam}_\rho(E)\big) \subseteq \Gamma_\kappa(x), \qquad \forall\, x_0, x \in E. \tag{6.26}$$

Consequently, whenever E is bounded there exists $C \in (0,\infty)$ such that for each $x_0 \in E$ one has

$$\mathcal{T}_\kappa(A) \subseteq B_{\rho_\#}\big(x_0, C\,\mathrm{diam}_\rho(E)\big), \qquad \forall\, A \text{ proper subset of } E. \tag{6.27}$$

Proof. With the exception of the first part of *(viii)*, these are direct consequences of definitions and the fact that both $\rho_\#(\cdot,\cdot)$ and $\delta_E(\cdot)$ are continuous functions. The remaining portion of the proof consists of a verification of (6.24). To get started, let x be an arbitrary point in $\mathcal{T}_\kappa(\mathcal{O})$. This places x in $\mathscr{X} \setminus E$ which, given that E is closed in $(\mathscr{X}, \tau_\rho)$, means that x does not belong to $\overline{\mathcal{O}} \subseteq E$. In particular, $\mathrm{dist}_{\rho_\#}(x, \mathcal{O}) > 0$. Going further, assume that some small $\varepsilon > 0$ has been fixed. The above discussion then shows that it is possible to pick a point $y \in \mathcal{O}$ with the property that

$$\rho_\#(x,y) < (1+\varepsilon)\,\mathrm{dist}_{\rho_\#}(x,\mathcal{O}). \tag{6.28}$$

Then there exists an index $j \in J$ for which $y \in \Delta_j$ and we shall show that ε and C_o can be chosen so as to guarantee that

$$x \in B_\rho(x_j, C_o r_j). \tag{6.29}$$

Indeed, selecting a real number $\beta \in (0, (\log_2 C_\rho)^{-1}]$ and invoking (6.18) we may write

$$\begin{aligned}
\big[\rho_\#(x,y)\big]^\beta &< (1+\varepsilon)^\beta \big[\mathrm{dist}_{\rho_\#}(x,\mathcal{O})\big]^\beta \leq \Big(\frac{1+\varepsilon}{1+\kappa}\Big)^\beta \big[\mathrm{dist}_{\rho_\#}(x, E\setminus\mathcal{O})\big]^\beta \\
&= \Big(\frac{1+\varepsilon}{1+\kappa}\Big)^\beta \mathrm{dist}_{(\rho_\#)^\beta}(x, E\setminus\mathcal{O}) \\
&\leq \Big(\frac{1+\varepsilon}{1+\kappa}\Big)^\beta \Big(\big[\rho_\#(x,y)\big]^\beta + \mathrm{dist}_{(\rho_\#)^\beta}(y, E\setminus\mathcal{O})\Big) \\
&\leq \Big(\frac{1+\varepsilon}{1+\kappa}\Big)^\beta \Big(\big[\rho_\#(x,y)\big]^\beta + C r_j^\beta\Big),
\end{aligned} \tag{6.30}$$

where $C \in (0,\infty)$ depends only on the geometrically doubling character of E. The last step above uses Theorem 2.2 and the fact that y belongs to $\Delta_j = B_\rho(x_j, r_j) \cap E$, which is a Whitney ball for $\mathcal{O}$. Choosing $\varepsilon = \kappa/2$, this now yields (on account of the first inequality in (2.14))

$$\rho(x,y) \leq C_\rho^2\,\rho_\#(x,y) < C_\rho^2 C^{1/\beta}\Big(\frac{1+\kappa/2}{\big[(1+\kappa)^\beta - (1+\kappa/2)^\beta\big]^{1/\beta}}\Big) r_j =: C_{\kappa,\beta}\, r_j. \tag{6.31}$$

Hence, since $\rho(x_j, x) \leq C_\rho \max\{\rho(x_j,y), \rho(y,x)\} < C_\rho C_{\kappa,\beta} r_j$, the membership in (6.29) holds provided we take $C_o := C_\rho C_{\kappa,\beta}$ to begin with. This finishes the proof of (6.24). $\square$

Lemma 6.3. *Let $(\mathscr{X}, \rho)$ be a quasi-metric space, E a proper, nonempty, closed subset of $(\mathscr{X}, \tau_\rho)$, μ a Borel measure on $(\mathscr{X}, \tau_\rho)$ and σ a Borel measure on $(E, \tau_{\rho|_E})$. Let $\rho_\#$ be associated with ρ as in Theorem 2.2 and recall the constant $C_\rho \geq 1$ defined in (2.2). Then for each real number $\kappa > 0$ there holds*

$$E \cap B_{\rho_\#}\big(y_*, \epsilon\delta_E(y)\big) \subseteq \pi_y^\kappa \subseteq E \cap B_{\rho_\#}\big(y_*, C_\rho(1+\kappa)\delta_E(y)\big), \qquad \forall\, y \in \mathscr{X} \setminus E, \tag{6.32}$$

where the point y_* and the number ϵ satisfy

$$\begin{aligned} &y_* \in E \text{ and } \rho_\#(y,y_*) < (1+\eta)\delta_E(y) \text{ for some } \eta\in(0,\kappa), \text{ and} \\ &0<\epsilon<\big[(1+\kappa)^\beta-(1+\eta)^\beta\big]^{1/\beta} \text{ for some finite } \beta\in(0,(\log_2 C_\rho)^{-1}]. \end{aligned} \tag{6.33}$$

In particular, if $(E,\rho\big|_E,\sigma)$ is a space of homogeneous type and if $\kappa,\kappa'>0$ are two arbitrary real numbers, then

$$c_o^{-1}\,\sigma(\pi_y^\kappa)\le\sigma(\pi_y^{\kappa'})\le c_o\,\sigma(\pi_y^\kappa), \qquad \forall\, y\in\mathscr{X}\setminus E, \tag{6.34}$$

where $c_o:=C_\sigma(C_\rho^2/\epsilon)^{D_\sigma}(1+\min\{\kappa,\kappa'\})^{D_\sigma}$, with C_σ and D_σ the doubling constant and doubling order of σ.

Proof. Fix an arbitrary point $y\in\mathscr{X}\setminus E$ and let $y_*\in E$ and $\epsilon>0$ be as in (6.33). If $x\in E\cap B_{\rho_\#}(y_*,\epsilon\delta_E(y))$ then $\rho_\#(y_*,x)<\epsilon\delta_E(y)$ forcing (recall from Theorem 2.2 that $(\rho_\#)^\beta$ is a genuine distance)

$$\rho_\#(x,y)^\beta\le\rho_\#(x,y_*)^\beta+\rho_\#(y_*,y)^\beta<\epsilon^\beta\delta_E(y)^\beta+(1+\eta)^\beta\delta_E(y)^\beta<(1+\kappa)^\beta\delta_E(y)^\beta. \tag{6.35}$$

Thus $x\in\pi_y^\kappa$, which proves the first inclusion in (6.32). Going further, given a point $x\in\pi_y^\kappa$ it follows that $\rho_\#(x,y)<(1+\kappa)\delta_E(y)$, hence

$$\rho_\#(x,y_*)\le C_{\rho_\#}\max\{\rho_\#(x,y),\rho_\#(y,y_*)\}<C_{\rho_\#}(1+\kappa)\delta_E(y)\le C_\rho(1+\kappa)\delta_E(y), \tag{6.36}$$

proving the second inclusion in (6.32).

Suppose now that $(E,\rho\big|_E,\sigma)$ is a space of homogeneous type and let $\kappa,\kappa'>0$ be given. Assume first that $\kappa\le\kappa'$. Choose y_* and ϵ as in (6.33). Then (6.32) holds both as written and with κ replaced by κ'. When combined with (2.37), this yields

$$c_1^{-1}\,\sigma(\pi_y^\kappa)\le\sigma(\pi_y^{\kappa'})\le c_1\,\sigma(\pi_y^\kappa), \qquad \forall\, y\in\mathscr{X}\setminus E, \tag{6.37}$$

where $c_1:=C_{\sigma,\rho_\#}\Big(\frac{C_\rho(1+\kappa)}{\epsilon}\Big)^{D_\sigma}$ with $C_{\sigma,\rho_\#}$ and D_σ being the constants associated with σ and $\rho_\#$ as in (2.37). In particular, since $C_{\sigma,\rho_\#}=C_\sigma(C_{\rho_\#}\widetilde{C}_{\rho_\#})^{D_\sigma}\le C_\sigma(C_\rho)^{D_\sigma}$, it follows that $c_1\le C_\sigma(C_\rho^2/\epsilon)^{D_\sigma}(1+\kappa)^{D_\sigma}$. If $\kappa'<\kappa$ the same reasoning yields inequalities similar to (6.37), this time with c_1 replaced by the constant $c_2:=C_{\sigma,\rho_\#}\Big(\frac{C_\rho(1+\kappa')}{\epsilon}\Big)^{D_\sigma}\le C_\sigma(C_\rho^2/\epsilon)^{D_\sigma}(1+\kappa')^{D_\sigma}$. All these now immediately yield (6.34). $\square$

Moving on, assume now that (E,ρ,σ) is a space of homogeneous type and let $\rho_\#$ be associated with ρ as in Theorem 2.2 in this context. Then for each σ-measurable set $A\subseteq E$ and each $\gamma\in(0,1)$, define the set of γ-`density points`, relative to A, as

$$A_\gamma^*:=\Big\{x\in E:\,\inf_{r>0}\Big[\frac{\sigma\big(B_{\rho_\#}(x,r)\cap A\big)}{\sigma\big(B_{\rho_\#}(x,r)\big)}\Big]\ge\gamma\Big\}. \tag{6.38}$$

In particular, from this definition it follows that

$$\inf_{x\in A_\gamma^*}\Big[\inf_{r>0}\frac{\sigma\big(B_{\rho_\#}(x,r)\cap A\big)}{\sigma\big(B_{\rho_\#}(x,r)\big)}\Big]\ge\gamma. \tag{6.39}$$

Some basic properties of the sets of density points in the setting of spaces of homogeneous type are collected below.

Proposition 6.4. *Let (E,ρ,σ) be a space of homogeneous type, $\rho_\#$ the regularization of ρ as in Theorem 2.2, $\gamma\in(0,1)$ and $A\subseteq E$ a σ-measurable set. Then the following properties hold:*

(1) $E\setminus A^*_\gamma=\big\{x\in E:\,M_E\big(\mathbf{1}_{E\setminus A}\big)(x)>1-\gamma\big\}$*, where M_E is the Hardy-Littlewood maximal operator on E (cf.* (2.103)*).*

(2) A^*_γ *is closed subset of* (E,τ_ρ)*.*

(3) $\sigma\big(E\setminus A^*_\gamma\big)\leq\frac{C}{1-\gamma}\sigma(E\setminus A)$*.*

(4) If A is closed (in τ_ρ), then $A^*_\gamma\subseteq A$. In particular, in this case, $\sigma(E\setminus A^*_\gamma)\approx\sigma(E\setminus A)$.

(5) For each $\lambda>0$ there exist $\gamma(\lambda)\in(0,1)$ and $c(\lambda)>0$ such that if $\gamma(\lambda)\leq\gamma<1$ then

$$\inf_{x\in E}\Big[\inf_{r>\mathrm{dist}_{\rho_\#}(x,A^*_\gamma)}\frac{\sigma\big(B_{\rho_\#}(x,\lambda r)\cap A\big)}{\sigma\big(B_{\rho_\#}(x,r)\big)}\Big]\geq c(\lambda). \tag{6.40}$$

(6) If the measure σ is Borel regular, then $\sigma(A^*_\gamma\setminus A)=0$.

(7) If $\widetilde{A}$ is σ-measurable set such that $A\subseteq\widetilde{A}\subseteq E$, then $A^*_\gamma\subseteq(\widetilde{A})^*_\gamma$.

The remarkable aspect of *(3)*-*(4)* above is that whenever A is a closed subset of (E,τ_ρ) then in a measure-theoretic sense the size of both sets, A^*_γ and $E\setminus A^*_\gamma$, may be controlled in terms of sets A and $E\setminus A$, respectively (as opposed to point-set theory). The typical application of Proposition 6.4 is in estimating the measure of a σ-measurable set $F\subseteq E$ by writing

$$\sigma(F)=\sigma(F\cap A^*_\gamma)+\sigma\big(F\cap(E\setminus A^*_\gamma)\big)\leq\sigma(F\cap A^*_\gamma)+\frac{C}{1-\gamma}\sigma(E\setminus A). \tag{6.41}$$

Proof of Proposition 6.4. Starting with (6.38) we may write

$$\begin{aligned} E\setminus A^*_\gamma &= \Big\{x\in E:\,\exists\,r>0\text{ such that }\frac{\sigma\big(B_{\rho_\#}(x,r)\cap A\big)}{\sigma\big(B_{\rho_\#}(x,r)\big)}<\gamma\Big\}\\ &= \Big\{x\in E:\,\exists\,r>0\text{ such that }\frac{\sigma\big(B_{\rho_\#}(x,r)\cap(E\setminus A)\big)}{\sigma\big(B_{\rho_\#}(x,r)\big)}>1-\gamma\Big\}\\ &= \Big\{x\in E:\,\sup_{r>0}\Big(⨍_{B_{\rho_\#}(x,r)}\mathbf{1}_{E\setminus A}\,d\sigma\Big)>1-\gamma\Big\}\\ &= \Big\{x\in E:\,\sup_{0<r\leq\mathrm{diam}_{\rho_\#}(E)}\Big(⨍_{B_{\rho_\#}(x,r)}\mathbf{1}_{E\setminus A}\,d\sigma\Big)>1-\gamma\Big\}\\ &= \Big\{x\in E:\,M_E\big(\mathbf{1}_{E\setminus A}\big)(x)>1-\gamma\Big\}, \end{aligned} \tag{6.42}$$

proving *(1)*. We now make the claim that

$$\text{the function } M_E\big(\mathbf{1}_{E\setminus A}\big):(E,\tau_\rho)\to[0,\infty] \text{ is lower semi-continuous.} \tag{6.43}$$

To prove this claim, we note that since the pointwise supremum of an arbitrary family of real-valued, lower semi-continuous functions defined on E is itself lower semi-continuous, it suffices to show that

$$\begin{array}{l}\text{for every } \sigma\text{-measurable set } F\subseteq E, \text{ the function } \ f:(E,\tau_\rho)\to[0,\infty)\\ \text{given by } \ f(x):=\sigma\big(B_{\rho_\#}(x,r)\cap F\big)\quad\forall\,x\in E, \ \text{ is lower semi-continuous.}\end{array}\tag{6.44}$$

To this end, fix $x_o\in E$ arbitrary. The crux of the matter is the fact that our choice of the quasi-distance ensures that if $\{x_j\}_{j\in\mathbb{N}}$ is a sequence of points in E with the property that $x_j\to x_o$ as $j\to\infty$, with convergence understood in the (metrizable) topology τ_ρ, then

$$\liminf_{j\to\infty}\mathbf{1}_{B_{\rho_\#}(x_j,r)}(y)\geq\mathbf{1}_{B_{\rho_\#}(x_o,r)}(y),\qquad\forall\,y\in E,\tag{6.45}$$

as is easily verified by analyzing the cases $y\in B_{\rho_\#}(x_o,r)$ and $y\in E\setminus B_{\rho_\#}(x_o,r)$. In turn, based on this and Fatou's lemma we may then estimate

$$\begin{aligned}f(x_o)&=\sigma\big(B_{\rho_\#}(x_o,r)\cap F\big)=\int_F\mathbf{1}_{B_{\rho_\#}(x_o,r)}(y)\,d\sigma(y)\\&\leq\int_F\liminf_{j\to\infty}\mathbf{1}_{B_{\rho_\#}(x_j,r)}(y)\,d\sigma(y)\leq\liminf_{j\to\infty}\int_F\mathbf{1}_{B_{\rho_\#}(x_j,r)}(y)\,d\sigma(y)\\&=\liminf_{j\to\infty}\sigma\big(B_{\rho_\#}(x_j,r)\cap F\big)=\liminf_{j\to\infty}f(x_j).\end{aligned}\tag{6.46}$$

This establishes (6.44), thus finishing the proof of (6.43).

Moving on, (6.43) implies that the last set in (6.42) is open (in τ_ρ), hence *(2)* holds true. Also, by combining *(1)* with the weak-$(1,1)$ boundedness of M_E (recall that we are assuming that (E,ρ,σ) is a space of homogeneous type), we obtain

$$\sigma\big(E\setminus A^*_\gamma\big)\leq\frac{C}{1-\gamma}\|\mathbf{1}_{E\setminus A}\|_{L^1(E,\sigma)}=\frac{C}{1-\gamma}\sigma(E\setminus A).\tag{6.47}$$

Hence the inequality in *(3)* is proved.

Suppose now that A is a closed subset of (E,τ_ρ). Then $E\setminus A$ is open, so if $x\in E\setminus A$ then there exists $r>0$ such that $B_{\rho_\#}(x,r)\subseteq E\setminus A$. Consequently, $\frac{\sigma\left(B_{\rho_\#}(x,r)\cap A)\right)}{\sigma\left(B_{\rho_\#}(x,r)\right)}=0<\gamma$, thus $x\notin A^*_\gamma$. This shows that $A^*_\gamma\subseteq A$, hence $\sigma(E\setminus A)\leq\sigma(E\setminus A^*_\gamma)$. Combining these with what we proved in *(3)* finishes the proof of *(4)*.

Turning to the proof of *(5)*, fix some $\lambda>0$ and $x\in E$, arbitrary, and select $r>0$ such that

$$\mathrm{dist}_{\rho_\#}(x,A^*_\gamma)<r.\tag{6.48}$$

Then there exists $x_0\in A^*_\gamma$ such that $\rho_\#(x,x_0)<r$, which forces

$$B_{\rho_\#}(x,\lambda r)\subseteq B_{\rho_\#}(x_0,C_{\rho_\#}(1+\lambda)r)\subseteq B_{\rho_\#}(x,C^2_{\rho_\#}(1+\lambda)r).\tag{6.49}$$

Consequently, since $x_0\in A^*_\gamma$ we obtain

$$\begin{aligned}\gamma\sigma\big(B_{\rho_\#}&(x_0,C_{\rho_\#}(1+\lambda)r)\big)\leq\sigma\big(B_{\rho_\#}(x_0,C_{\rho_\#}(1+\lambda)r)\cap A\big)\\&\leq\sigma\big(B_{\rho_\#}(x_0,C_{\rho_\#}(1+\lambda)r)\setminus B_{\rho_\#}(x,\lambda r)\big)+\sigma\big(B_{\rho_\#}(x,\lambda r)\cap A\big)\\&=\sigma\big(B_{\rho_\#}(x_0,C_{\rho_\#}(1+\lambda)r)\big)-\sigma\big(B_{\rho_\#}(x,\lambda r)\big)+\sigma\big(B_{\rho_\#}(x,\lambda r)\cap A\big),\end{aligned}\tag{6.50}$$

which further implies that

$$\sigma\big(B_{\rho_\#}(x,\lambda r)\big)-(1-\gamma)\sigma\big(B_{\rho_\#}(x_0,C_{\rho_\#}(1+\lambda)r)\big)\leq\sigma\big(B_{\rho_\#}(x,\lambda r)\cap A\big). \tag{6.51}$$

Recalling the second inclusion in (6.49) and (2.37), we obtain

$$\begin{aligned}\sigma\big(B_{\rho_\#}(x_0,C_{\rho_\#}(1+\lambda)r)\big) &\leq \sigma\big(B_{\rho_\#}(x_0,C^2_{\rho_\#}(1+\lambda)r)\big)\\ &\leq C_{\sigma,\rho_\#}\big(\tfrac{C^2_{\rho_\#}(1+\lambda)}{\lambda}\big)^{D_\sigma}\sigma\big(B_{\rho_\#}(x,\lambda r)\big),\end{aligned} \tag{6.52}$$

where $C_{\sigma,\rho_\#}$, D_σ are associated with σ, $\rho_\#$ as in (2.37). Together, (6.51) and (6.52) yield

$$\sigma\big(B_{\rho_\#}(x,\lambda r)\big)\Big[1-C_{\sigma,\rho_\#}(1-\gamma)\Big(\tfrac{C^2_{\rho_\#}(1+\lambda)}{\lambda}\Big)^{D_\sigma}\Big]\leq\sigma\big(B_{\rho_\#}(x,\lambda r)\cap A\big). \tag{6.53}$$

Also, from (2.37) we have that if $\lambda\in(0,1)$ then $\sigma\big(B_{\rho_\#}(x,r)\big)\leq C_{\sigma,\rho_\#}\lambda^{-D_\sigma}\sigma\big(B_{\rho_\#}(x,\lambda r)\big)$, thus

$$\sigma\big(B_{\rho_\#}(x,\lambda r)\big)\geq\min\Big\{1,\frac{\lambda^{D_\sigma}}{C_{\sigma,\rho_\#}}\Big\}\sigma\big(B_{\rho_\#}(x,r)\big),\qquad\forall\,\lambda>0. \tag{6.54}$$

If we now we choose

$$\gamma(\lambda):=1-\frac{1}{2C_{\sigma,\rho_\#}}\Big(\frac{\lambda}{C^2_{\rho_\#}(1+\lambda)}\Big)^{D_\sigma}\in(0,1)\quad\text{ and }\quad c(\lambda):=\frac{1}{2}\min\Big\{1,\frac{\lambda^{D_\sigma}}{C_{\sigma,\rho_\#}}\Big\}>0, \tag{6.55}$$

then (6.53) and (6.54) imply

$$\sigma\big(B_{\rho_\#}(x,\lambda r)\cap A\big)\geq\frac{1}{2}\sigma\big(B_{\rho_\#}(x,\lambda r)\big)\geq c(\lambda)\sigma\big(B_{\rho_\#}(x,r)\big),\qquad\forall\,\gamma\in\big[\gamma(\lambda),1\big). \tag{6.56}$$

This proves *(5)*.

If σ is Borel-regular, then Lebesgue's Differentiation Theorem holds in the current setting. Hence, there exists a set $F\subseteq E$ with $\sigma(F)=0$ and such that

$$\lim_{r\to 0^+}\Big(⨍_{B_{\rho_\#}(x,r)}\mathbf{1}_A\,d\sigma\Big)=\mathbf{1}_A(x),\qquad\forall\,x\in E\setminus F. \tag{6.57}$$

In particular, for every $x\in A^*_\gamma\setminus F$ we have $\mathbf{1}_A(x)=\lim\limits_{r\to 0^+}\Big[\frac{\sigma\big(B_{\rho_\#}(x,r)\cap A\big)}{\sigma\big(B_{\rho_\#}(x,r)\big)}\Big]\geq\gamma>0$, which implies that $A^*_\gamma\setminus F\subseteq A$, thus $A^*_\gamma\setminus A\subseteq F$. Consequently, since $A^*_\gamma\setminus A$ is σ-measurable, we obtain that $\sigma(A^*_\gamma\setminus A)=0$, proving *(6)*. Finally, the statement in *(7)* is an immediate consequence of (6.38). This concludes the proof of the proposition. $\square$

We continue to state and prove auxiliary lemmas in preparation for dealing with Theorem 6.8, advertised earlier. To state the lemma below, recall the region $\mathcal{F}_\kappa(A)$ from (6.16).

Lemma 6.5. *Let $(\mathscr{X},\rho)$ be a quasi-metric space, μ a Borel measure on $(\mathscr{X},\tau_\rho)$, E a proper, nonempty, closed subset of $(\mathscr{X},\tau_\rho)$ and σ a Borel measure on $(E,\tau_{\rho|_E})$ such that $(E,\rho\big|_E,\sigma)$ is a space of homogeneous type. If $u:\mathscr{X}\setminus E\to[0,\infty]$ is μ-measurable, then for every $\kappa>0$ and every σ-measurable set $A\subseteq E$, one has*

$$\begin{aligned}\int_A\Big(\int_{\Gamma_\kappa(x)}u(y)\,d\mu(y)\Big)\,d\sigma(x) &= \int_{\mathscr{X}\setminus E}u(y)\sigma\big(A\cap\pi^\kappa_y\big)\,d\mu(y)\\ &= \int_{\mathcal{F}_\kappa(A)}u(y)\sigma\big(A\cap\pi^\kappa_y\big)\,d\mu(y).\end{aligned} \tag{6.58}$$

Proof. By Fubini's Theorem (and (6.9)), we have

$$\begin{aligned}\int_A \Big(\int_{\Gamma_\kappa(x)} u(y)\,d\mu(y)\Big)\,d\sigma(x) &= \int_{\mathscr{X}\setminus E} u(y)\Big(\int_A \mathbf{1}_{\pi_y^\kappa}(x)\,d\sigma(x)\Big)\,d\mu(y)\\ &= \int_{\mathscr{X}\setminus E} u(y)\sigma\big(A\cap\pi_y^\kappa\big)\,d\mu(y), \end{aligned}\tag{6.59}$$

proving the first equality in (6.58). The second equality in (6.58) follows from (6.59) and the fact that if $y\in\mathscr{X}\setminus E$ and $A\cap\pi_y^\kappa\neq\emptyset$ then $y\in\mathcal{F}_\kappa(A)$. $\square$

Lemma 6.6. *Let $(\mathscr{X},\rho)$ be a quasi-metric space, μ a Borel measure on $(\mathscr{X},\tau_\rho)$, E a proper, nonempty, closed subset of $(\mathscr{X},\tau_\rho)$, and σ a Borel measure on $(E,\tau_{\rho|_E})$ such that $(E,\rho\big|_E,\sigma)$ is a space of homogeneous type. Fix two arbitrary numbers $\kappa,\kappa'>0$. Then there exist $\gamma\in(0,1)$ and a finite constant $C>0$ such that for every σ-measurable set $A\subseteq E$ there holds*

$$\int_{A_\gamma^*}\Big(\int_{\Gamma_\kappa(x)} u(y)\,d\mu(y)\Big)\,d\sigma(x)\leq C\int_A\Big(\int_{\Gamma_{\kappa'}(x)} u(y)\,d\mu(y)\Big)\,d\sigma(x)\tag{6.60}$$

for every function $u:\mathscr{X}\setminus E\to[0,\infty]$ which is μ-measurable.

Proof. Recall the notation introduced in (2.19). We claim that

$$\begin{gathered}\text{for every }\ \kappa,\kappa'>0\ \text{ there exist }\gamma\in(0,1)\ \text{ and }\ c>0\ \text{ such that}\\ \sigma\big(A\cap\pi_y^{\kappa'}\big)\geq c\,\sigma\big(A_\gamma^*\cap\pi_y^\kappa\big)\quad\forall\,A\subseteq E\ \sigma\text{-measurable and }\forall\,y\in\mathcal{F}_\kappa(A_\gamma^*).\end{gathered}\tag{6.61}$$

Assuming this claim for now, let $\kappa,\kappa'>0$ be arbitrary and let γ and $c>0$ be as in (6.61). Then, if A and u satisfy the hypotheses of the proposition, starting with (6.58) and using the fact that $\mathcal{F}_\kappa(A_\gamma^*)\subseteq\mathscr{X}\setminus E$ (itself a trivial consequence of (6.2)), we may write

$$\begin{aligned}\int_A\Big(\int_{\Gamma_{\kappa'}(x)} u(y)\,d\mu(y)\Big)\,d\sigma(x) &= \int_{\mathscr{X}\setminus E} u(y)\sigma\big(A\cap\pi_y^{\kappa'}\big)\,d\mu(y)\\ &\geq \int_{\mathcal{F}_\kappa(A_\gamma^*)} u(y)\sigma\big(A\cap\pi_y^{\kappa'}\big)\,d\mu(y)\\ &\geq c\int_{\mathcal{F}_\kappa(A_\gamma^*)} u(y)\sigma\big(A_\gamma^*\cap\pi_y^{\kappa}\big)\,d\mu(y)\\ &= c\int_{A_\gamma^*}\Big(\int_{\Gamma_\kappa(x)} u(y)\,d\mu(y)\Big)\,d\sigma(x),\end{aligned}\tag{6.62}$$

where for the last equality in (6.62) we applied Lemma 6.5 with A_γ^* in place of A. Hence, to finish the proof of the proposition we are left with showing (6.61).

Suppose $\kappa,\kappa'>0$ are fixed and pick some $\gamma\in(0,1)$, to be made precise later. Also, fix $\eta\in(0,\min\{\kappa,\kappa'\})$ and for each $y\in\mathcal{F}_\kappa(A_\gamma^*)$ choose $y_*\in E$ and $\epsilon>0$ as in (6.33) (for η as just indicated). Then y_* satisfies the conditions in (6.33) corresponding to both κ and κ'. As such, Lemma 6.3 implies that the inclusions in (6.32) hold for both κ and κ'.

The fact that $y\in\mathcal{F}_\kappa(A_\gamma^*)$ entails $\pi_y^\kappa\cap A_\gamma^*\neq\emptyset$ which, when combined with (6.32), implies $B_{\rho_\#}\big(y_*,C_\rho(1+\kappa)\delta_E(y)\big)\cap A_\gamma^*\neq\emptyset$ hence, further, $\mathrm{dist}_{\rho_\#}(y_*,A_\gamma^*)<C_\rho(1+\kappa)\delta_E(y)$. Now, *(5)*

in Proposition 6.4 invoked with $\lambda := \frac{\epsilon}{C_\rho(1+\kappa)}$, $x := y_*$ and $r := C_\rho(1+\kappa)\delta_E(y)$, guarantees the existence of some $\gamma_0 = \gamma_0(\lambda) \in (0,1)$ with the property that

$$\frac{\sigma\big(B_{\rho_\#}(y_*, \epsilon\delta_E(y)) \cap A\big)}{\sigma\big(B_{\rho_\#}(y_*, C_\rho(1+\kappa)\delta_E(y))\big)} \geq c = c(\kappa) > 0 \quad \text{ if } \gamma \in (\gamma_0, 1). \tag{6.63}$$

Hence, if we select $\gamma \in (\gamma_0, 1)$ to begin with, the estimate in (6.63) in concert with (6.32) implies

$$\sigma\big(B_{\rho_\#}(y_*, \epsilon\delta_E(y)) \cap A\big) \geq c\,\sigma\big(B_{\rho_\#}(y_*, C_\rho(1+\kappa)\delta_E(y))\big) \geq c\,\sigma\big(\pi_y^\kappa\big) \geq c\,\sigma\big(A_\gamma^* \cap \pi_y^\kappa\big). \tag{6.64}$$

Since (6.32) also holds with κ replaced by κ', we obtain from this and (6.64) that

$$\sigma\big(A \cap \pi_y^{\kappa'}\big) \geq \sigma\big(B_{\rho_\#}(y_*, \epsilon\delta_E(y)) \cap A\big) \geq c\,\sigma\big(A_\gamma^* \cap \pi_y^\kappa\big). \tag{6.65}$$

This completes the proof of (6.61) and, with it, the proof of the lemma. $\square$

Lemma 6.7. *Let $(\mathscr{X}, \rho)$ be a quasi-metric space, μ a Borel measure on $(\mathscr{X}, \tau_\rho)$, E a proper, nonempty, closed subset of $(\mathscr{X}, \tau_\rho)$, and σ a Borel measure on $(E, \tau_{\rho|_E})$ such that $(E, \rho\big|_E, \sigma)$ is a space of homogeneous type. Then for every $\kappa, \kappa' > 0$ there exists a constant $C \in (0, \infty)$ such that*

$$\int_E \Big(\int_{\Gamma_\kappa(x)} u(y)\,d\mu(y)\Big) f(x)\,d\sigma(x) \leq C \int_E \Big(\int_{\Gamma_{\kappa'}(x)} u(y)\,d\mu(y)\Big)(M_E f)(x)\,d\sigma(x) \tag{6.66}$$

for every function $u : \mathscr{X} \setminus E \to [0, \infty]$ that is μ-measurable, and every function $f : E \to [0, \infty]$ that is σ-measurable.

Proof. Based on Fubini's Theorem (and (6.9)), we may write

$$\begin{aligned} \int_E \Big(\int_{\Gamma_\kappa(x)} u(y)\,d\mu(y)\Big) f(x)\,d\sigma(x) &= \int_{\mathscr{X}\setminus E} u(y) \Big(\int_E \mathbf{1}_{\pi_y^\kappa}(x) f(x)\,d\sigma(x)\Big)\,d\mu(y) \\ &= \int_{\mathscr{X}\setminus E} u(y)\sigma\big(\pi_y^\kappa\big) \Big(\fint_{\pi_y^\kappa} f\,d\sigma\Big)\,d\mu(y), \end{aligned} \tag{6.67}$$

as well as

$$\int_E \Big(\int_{\Gamma_{\kappa'}(x)} u(y)\,d\mu(y)\Big)(M_E f)(x)\,d\sigma(x) = \int_{\mathscr{X}\setminus E} u(y)\sigma\big(\pi_y^\kappa\big) \Big(\fint_{\pi_y^{\kappa'}} M_E f\,d\sigma\Big)\,d\mu(y). \tag{6.68}$$

Hence, in order to conclude (6.66), in light of (6.67), (6.68), and (6.34), it suffices to show that there exists a constant $C_1 \in (0, \infty)$ such that

$$\fint_{\pi_y^\kappa} f\,d\sigma \leq C_1 \fint_{\pi_y^{\kappa'}} M_E f\,d\sigma \qquad \text{for every } y \in \mathscr{X} \setminus E. \tag{6.69}$$

To this end, fix some $y \in \mathscr{X} \setminus E$ and let $y_* \in E$, $\epsilon > 0$ be such that (6.33) holds for some $\eta \in (0, \min\{\kappa, \kappa'\})$. Then (6.32) holds when written both for κ and κ'. In particular, for each $z \in \pi_y^{\kappa'}$ we have $\rho_\#(z, y_*) < C_\rho(1+\kappa')\delta_E(y)$ and, consequently,

$$\begin{aligned} B_{\rho_\#}\big(y_*, C_\rho(1+\kappa)\delta_E(y)\big) &\subseteq B_{\rho_\#}\big(z, C_\rho^2(1+\max\{\kappa,\kappa'\})\delta_E(y)\big) \\ &\subseteq B_{\rho_\#}\big(y_*, C_\rho^3(1+\max\{\kappa,\kappa'\})\delta_E(y)\big), \qquad \forall\, z \in \pi_y^{\kappa'}. \end{aligned} \tag{6.70}$$

Making now use of (6.32), (6.70) and (2.37), we obtain

$$\begin{aligned}
\fint_{\pi_y^\kappa} f\,d\sigma &\leq \frac{1}{\sigma(B_{\rho_\#}(y_*,\epsilon\delta_E(y)))}\int_{B_{\rho_\#}\left(y_*,C_\rho(1+\kappa)\delta_E(y)\right)} f\,d\sigma \\
&\leq C_{\sigma,\rho_\#}\big(C_\rho^3\epsilon^{-1}(1+\max\{\kappa,\kappa'\})\big)^{D_\sigma}\fint_{B_{\rho_\#}(z,C_\rho^2(1+\max\{\kappa,\kappa'\})\delta_E(y))} f\,d\sigma \\
&\leq C_{\sigma,\rho_\#}\big(C_\rho^3\epsilon^{-1}(1+\max\{\kappa,\kappa'\})\big)^{D_\sigma} M_E f(z), \qquad \forall\, z\in\pi_y^{\kappa'},
\end{aligned}\tag{6.71}$$

where $C_{\sigma,\rho_\#}$, D_σ are the constants associated with σ, $\rho_\#$ as in (2.37). Thus, if we now set $C_1 := C_{\sigma,\rho_\#}\big(C_\rho^3\epsilon^{-1}(1+\max\{\kappa,\kappa'\})\big)^{D_\sigma}$ then

$$\fint_{\pi_y^\kappa} f\,d\sigma \leq C_1 \inf_{z\in\pi_y^{\kappa'}}[M_E f(z)] \leq C_1 \fint_{\pi_y^{\kappa'}} M_E f\,d\sigma, \tag{6.72}$$

proving (6.69), and finishing the proof of the lemma. $\square$

We are now prepared to state and prove the following equivalence result for the quasi-norms of the mixed norm spaces associated with different apertures (of the nontangential approach regions).

Theorem 6.8. *Let $(\mathscr{X},\rho)$ be a quasi-metric space, μ a Borel measure on $(\mathscr{X},\tau_\rho)$, E a proper, nonempty, closed subset of $(\mathscr{X},\tau_\rho)$, and σ a Borel measure on $(E,\tau_{\rho|_E})$ such that $(E,\rho\big|_E,\sigma)$ is a space of homogeneous type. Also, fix two indices $p,q\in(0,\infty]$ with the convention that $q=\infty$ if $p=\infty$. Then for each $\kappa,\kappa'>0$ there holds*

$$\|u\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)} \approx \|u\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa')}, \tag{6.73}$$

uniformly for μ-measurable functions $u:\mathscr{X}\setminus E\to\overline{\mathbb{R}}$.

Hence, in particular, for each $p,q\in(0,\infty)$, there holds

$$\Big(\int_E\Big[\int_{\Gamma_\kappa(x)}|u(y)|^q\,d\mu(y)\Big]^{p/q}d\sigma(x)\Big)^{1/p} \approx \Big(\int_E\Big[\int_{\Gamma_{\kappa'}(x)}|u(y)|^q\,d\mu(y)\Big]^{p/q}d\sigma(x)\Big)^{1/p}, \tag{6.74}$$

uniformly for μ-measurable functions $u:\mathscr{X}\setminus E\to\overline{\mathbb{R}}$.

Before presenting the proof of this theorem we shall comment on the nature of the limiting case $p=\infty$, $q\in(0,\infty)$ of (6.74). This clarifies the comment at the bottom of page 183 in [71].

Remark 6.9. *In the context of Theorem 6.8, if $q\in(0,\infty)$, in general it is not true that*

$$\sup_{x\in E}\Big(\int_{\Gamma_\kappa(x)}|u(y)|^q\,d\mu(y)\Big)^{1/q} \approx \sup_{x\in E}\Big(\int_{\Gamma_{\kappa'}(x)}|u(y)|^q\,d\mu(y)\Big)^{1/q}. \tag{6.75}$$

To see that this equivalence might fail, consider the case when $\mathscr{X}:=\mathbb{R}^2$, $E:=\mathbb{R}\equiv\partial\mathbb{R}^2_+$, and take $\kappa:=\sqrt{2}$, $\kappa'\in(0,\sqrt{2})$. Also, without loss of generality, assume that $q=1$ and consider $u:\mathscr{X}\setminus E\to\overline{\mathbb{R}}$ given by

$$u(x,y) := \begin{cases} x^{-1} & \textit{if } x>0 \textit{ and } x<y<x+1,\\ 0 & \textit{otherwise.}\end{cases} \tag{6.76}$$

Then

$$\sup_{z\in\mathbb{R}}\Big(\int_{\Gamma_\kappa(z)}|u(x,y)|\,dxdy\Big)=\int_{|x|<y}|u(x,y)|\,dxdy=\int_{0<x<y<x+1}x^{-1}\,dxdy=\infty, \tag{6.77}$$

whereas for each $z\in(0,\infty)$, *elementary geometry gives that*

$$\int_{\Gamma_{\kappa'}(z)}|u(x,y)|\,dxdy\le Cz^{-1}\cdot\mathrm{Area}\{(x,y)\in\Gamma_{\kappa'}(z):\,0<x<y<x+1\}\le C, \tag{6.78}$$

for some $C=C(\kappa')\in(0,\infty)$. *This shows that* $\sup_{z\in\mathbb{R}}\Big(\int_{\Gamma_{\kappa'}(z)}|u(x,y)|\,dxdy\Big)<\infty$, *hence* (6.75) *fails in this case.*

We now turn to the

Proof of Theorem 6.8. Let the real numbers $\kappa,\kappa'>0$ be arbitrary and fixed. Then, recalling (6.7), it follows that the equivalence in (6.73) is proved once we show that there exists a finite constant $C=C(\kappa,\kappa')>0$ such that for every μ-measurable function $u:\mathscr{X}\setminus E\to\overline{\mathbb{R}}$ we have

$$\|\mathscr{A}_{q,\kappa'}u\|_{L^p(E,\sigma)}\le C\|\mathscr{A}_{q,\kappa}u\|_{L^p(E,\sigma)}, \tag{6.79}$$

with the understanding that, when $q=\infty$, the q-th power integral of u over nontangential approach regions is replaced by the nontangential maximal operator of u (cf. (6.13)). We proceed by dividing up the proof of (6.79) into a number of cases.

Case I: $0<p<q<\infty$. For $\lambda>0$ arbitrary define

$$A:=\big\{x\in E:\,(\mathscr{A}_{q,\kappa}u)(x)\le\lambda\big\}. \tag{6.80}$$

By (6.8) we have that A is closed in $(E,\tau_{\rho|_E})$, hence $A^*_\gamma\subseteq A$ for every $\gamma\in(0,1)$ by virtue of *(4)* in Proposition 6.4. Let $\gamma=\gamma(\kappa,\kappa')\in(0,1)$ be such that (6.60) holds. Then

$$\begin{aligned}\sigma\big(\{x\in E:\,(\mathscr{A}_{q,\kappa'}u)(x)>\lambda\}\big)&\le\sigma\big(E\setminus A^*_\gamma\big)+\sigma\big(\{x\in A^*_\gamma:\,(\mathscr{A}_{q,\kappa'}u)(x)>\lambda\}\big)\\&\le\frac{C}{1-\gamma}\sigma(E\setminus A)+\frac{1}{\lambda^q}\int_{A^*_\gamma}(\mathscr{A}_{q,\kappa'}u)(x)^q\,d\sigma(x)\\&\le\frac{C}{1-\gamma}\sigma\big(\{x\in E:\,(\mathscr{A}_{q,\kappa}u)(x)>\lambda\}\big)+\frac{C}{\lambda^q}\int_A(\mathscr{A}_{q,\kappa}u)(x)^q\,d\sigma(x).\end{aligned} \tag{6.81}$$

For the second inequality in (6.81) we used *(3)* in Proposition 6.4 and Tschebyshev's inequality, while for the last inequality we used (6.80) and (6.60) (with κ and κ' interchanged). Thus, if we multiply the inequality resulting from (6.81) by $p\lambda^{p-1}$ and then integrate in $\lambda\in(0,\infty)$, we obtain

$$\|\mathscr{A}_{q,\kappa'}u\|^p_{L^p(E,\sigma)}\le\frac{C}{1-\gamma}\|\mathscr{A}_{q,\kappa}u\|^p_{L^p(E,\sigma)}+C\int_0^\infty\lambda^{p-q-1}\Big(\int_{\{\mathscr{A}_{q,\kappa}u\le\lambda\}}(\mathscr{A}_{q,\kappa}u)^q\,d\sigma\Big)\,d\lambda. \tag{6.82}$$

By Fubini's Theorem, we further write

$$\begin{aligned}\int_0^\infty\lambda^{p-q-1}\Big(\int_{\{\mathscr{A}_{q,\kappa}u\le\lambda\}}(\mathscr{A}_{q,\kappa}u)^q\,d\sigma\Big)\,d\lambda&=\int_E\Big(\int_{(\mathscr{A}_{q,\kappa}u)(x)}^\infty\lambda^{p-q-1}\,d\lambda\Big)(\mathscr{A}_{q,\kappa}u)(x)^q\,d\sigma(x)\\&=(q-p)^{-1}\|\mathscr{A}_{q,\kappa}u\|^p_{L^p(E,\sigma)},\end{aligned} \tag{6.83}$$

given that we are currently assuming that $p<q$. In concert, (6.82) and (6.83) now yield (6.79) in the case when $q\in(0,\infty)$ and $0<p<q$.

Case II: $p=q\in(0,\infty)$. Combining (6.58) (corresponding to $A=E$ and applied twice) with (6.34) (applied for every $y\in\mathscr{X}\setminus E$), we obtain that

$$\begin{aligned}
\int_E\Big(\int_{\Gamma_{\kappa'}(x)}|u(y)|^p\,d\mu(y)\Big)\,d\sigma(x) &= \int_{\mathscr{X}\setminus E}|u(y)|^p\sigma(\pi_y^{\kappa'})\,d\mu(y)\\
&\approx \int_{\mathscr{X}\setminus E}|u(y)|^p\sigma(\pi_y^{\kappa})\,d\mu(y)\\
&= \int_E\Big(\int_{\Gamma_{\kappa}(x)}|u(y)|^p\,d\mu(y)\Big)\,d\sigma(x), \qquad (6.84)
\end{aligned}$$

and the desired conclusion follows.

Case III: $0<q<p<\infty$. Let $(p/q)'$ denote the Hölder conjugate of $p/q\in(1,\infty)$. By using Riesz's duality theorem for Lebesgue spaces, then Lemma 6.7 (with u replaced by $|u|^q$), and then Hölder's inequality, we may write

$$\begin{aligned}
\bigg(\int_E\Big(\int_{\Gamma_{\kappa}(x)}|u(y)|^q\,d\mu(y)\Big)^{p/q}\,d\sigma(x)\bigg)^{q/p} &= \Big\|\int_{\Gamma_{\kappa}(x)}|u|^q\,d\mu\Big\|_{L_x^{p/q}(E,\sigma)}\\
&= \sup_{\substack{f\in L^{(p/q)'}(E,\sigma)\\ f\geq 0,\ \|f\|_{L^{(p/q)'}(E,\sigma)}\leq 1}}\bigg[\int_E\Big(\int_{\Gamma_{\kappa}(x)}|u(y)|^q\,d\mu(y)\Big)f(x)\,d\sigma(x)\bigg]\\
&\leq C\sup_{\substack{f\in L^{(p/q)'}(E,\sigma)\\ f\geq 0,\ \|f\|_{L^{(p/q)'}(E,\sigma)}\leq 1}}\bigg[\int_E\Big(\int_{\Gamma_{\kappa'}(x)}|u(y)|^q\,d\mu(y)\Big)(M_E f)(x)\,d\sigma(x)\bigg]\\
&\leq C\sup_{\substack{f\in L^{(p/q)'}(E,\sigma)\\ f\geq 0,\ \|f\|_{L^{(p/q)'}(E,\sigma)}\leq 1}}\bigg[\bigg(\int_E\Big(\int_{\Gamma_{\kappa'}(x)}|u(y)|^q\,d\mu(y)\Big)^{p/q}\,d\sigma(x)\bigg)^{q/p}\times\\
&\qquad\qquad\times\Big(\int_E(M_E f)(x)^{(p/q)'}\,d\sigma(x)\Big)^{1/(p/q)'}\bigg]\\
&\leq C\bigg(\int_E\Big(\int_{\Gamma_{\kappa'}(x)}|u(y)|^q\,d\mu(y)\Big)^{p/q}\,d\sigma(x)\bigg)^{q/p}, \qquad (6.85)
\end{aligned}$$

where for the last inequality in (6.85) we used the boundedness of the maximal operator M_E on $L^r(E,\sigma)$ for $r:=(p/q)'\in(1,\infty)$. This completes the proof of (6.79) when $0<q<p<\infty$.

Case IV: $0<p<\infty$, $q=\infty$. Fix $\lambda>0$ and introduce (recall (6.13))

$$\mathcal{O}_\kappa:=\big\{x\in E:\,(\mathcal{N}_\kappa u)(x)>\lambda\big\},\qquad \mathcal{O}_{\kappa'}:=\big\{x\in E:\,(\mathcal{N}_{\kappa'}u)(x)>\lambda\big\}. \qquad (6.86)$$

Hence, the desired conclusion follows as soon as we show that there exists $C\in(0,\infty)$, independent of u and λ, with the property that $\sigma(\mathcal{O}_{\kappa'})\leq C\sigma(\mathcal{O}_\kappa)$. In turn, by virtue of *(3)* in

Proposition 6.4, this follows once we prove that that there exists $\gamma \in (0,1)$ such that

$$\mathcal{O}_{\kappa'} \subseteq E \setminus (E \setminus \mathcal{O}_\kappa)^*_\gamma. \tag{6.87}$$

To justify this inclusion, fix $\eta \in (0, \min\{\kappa, \kappa'\})$ and assume that $x \in \mathcal{O}_{\kappa'}$ is an arbitrary point. Then there exists $y \in \Gamma_{\kappa'}(x)$ for which $|u(y)| > \lambda$ and we select $y_* \in E$ and $\epsilon \in (0,1)$ as in (6.33) (for η as specified above). In particular, $\rho_\#(y, y_*) < (1+\eta)\delta_E(y)$. Observe from (6.32) and (6.86) that in this scenario we have

$$E \cap B_{\rho_\#}\big(y_*, \epsilon\delta_E(y)\big) \subseteq \pi^\kappa_y \subseteq \mathcal{O}_\kappa \tag{6.88}$$

and we also claim that

$$E \cap B_{\rho_\#}\big(y_*, \epsilon\delta_E(y)\big) \subseteq E \cap B_{\rho_\#}\big(x, C_\rho(1+\kappa')\delta_E(y)\big). \tag{6.89}$$

To see this, recall that $\epsilon \in (0,1)$ and note that if $z \in E$ satisfies $\rho_\#(z, y_*) < \delta_E(y)$ then

$$\begin{aligned}
\rho_\#(x,z) &\le C_\rho \max\{\rho_\#(x,y), \rho_\#(y,z)\} \\
&\le C_\rho \max\Big\{(1+\kappa')\delta_E(y),\, C_\rho \max\{\rho_\#(y,y_*), \rho_\#(y_*,z)\}\Big\} \\
&\le C_\rho \max\Big\{(1+\kappa')\delta_E(y),\, C_\rho \max\{(1+\alpha)\delta_E(y), \delta_E(y)\}\Big\} \\
&= C_\rho(1+\kappa')\delta_E(y),
\end{aligned} \tag{6.90}$$

proving (6.89). In concert, (6.88) and (6.89) yield

$$E \cap B_{\rho_\#}\big(y_*, \delta_E(y)\big) \subseteq \mathcal{O}_\kappa \cap B_{\rho_\#}\big(x, C_\rho(1+\kappa')\delta_E(y)\big). \tag{6.91}$$

Let us also observe that

$$\begin{aligned}
\rho_\#(x,y_*) &\le C_\rho \max\{\rho_\#(x,y), \rho_\#(y,y_*)\} \\
&\le C_\rho \max\Big\{(1+\kappa')\delta_E(y), (1+\eta)\delta_E(y)\Big\} = C_\rho(1+\kappa')\delta_E(y).
\end{aligned} \tag{6.92}$$

Then, for some sufficiently small $c \in (0,1)$ which depends only on κ, κ' and background geometrical characteristics, we may write

$$\frac{\sigma\Big(\mathcal{O}_\kappa \cap B_{\rho_\#}\big(x, C_\rho(1+\kappa')\delta_E(y)\big)\Big)}{\sigma\Big(E \cap B_{\rho_\#}\big(x, C_\rho(1+\kappa')\delta_E(y)\big)\Big)} \ge \frac{\sigma\Big(E \cap B_{\rho_\#}\big(y_*, \epsilon\delta_E(y)\big)\Big)}{\sigma\Big(E \cap B_{\rho_\#}\big(x, C_\rho(1+\kappa')\delta_E(y)\big)\Big)} \ge c, \tag{6.93}$$

where the first inequality follows from (6.91), while the second inequality is a consequence of (6.92) and the fact that $(E, \rho\big|_E, \sigma)$ is a space of homogeneous type (cf. (2.37)). In particular, if we set $r := C_\rho(1+\kappa')\delta_E(y)$, then

$$\frac{\sigma\big((E \setminus \mathcal{O}_\kappa) \cap B_{\rho_\#}(x,r)\big)}{\sigma\big(E \cap B_{\rho_\#}(x,r)\big)} \le 1 - c. \tag{6.94}$$

Thus, if we select γ such that $1 - c < \gamma < 1$, then (6.94) entails $x \notin (E \setminus \mathcal{O}_\kappa)^*_\gamma$. This proves the claim (6.87), and finishes the treatment of the current case.

Case V: $p = q = \infty$. In this case, the desired conclusion follows upon observing that if $u : \mathscr{X} \setminus E \to \overline{\mathbb{R}}$ is a μ-measurable function then

$$\Big\| E \ni x \mapsto \|u\|_{L^\infty(\Gamma_\kappa(x),\mu)} \Big\|_{L^\infty(E,\sigma)} = \|u\|_{L^\infty(\mathscr{X}\setminus E,\mu)}. \tag{6.95}$$

Indeed, the inequality $\big\|\|u\|_{L^\infty(\Gamma_\kappa(x),\mu)}\big\|_{L^\infty_x(E,\sigma)} \leq \|u\|_{L^\infty(\mathscr{X}\setminus E,\mu)}$ is a simple consequence of the fact that $\Gamma_\kappa(x) \subseteq \mathscr{X} \setminus E$ for each $x \in E$. In the opposite direction, if M denotes the left-hand side of (6.95), then there exists a σ-measurable set $F \subseteq E$ satisfying $\sigma(F) = 0$ and $\|u\|_{L^\infty(\Gamma_\kappa(x),\mu)} \leq M$ for every $x \in E \setminus F$. Since $(E, \rho|_E)$ is geometrically doubling, so is $E \setminus F$ when equipped with $\rho|_{E\setminus F}$, hence separable as a topological space. Consequently, given that $E \setminus F$ is dense in E, it follows that there exists a countable subset $A := \{x_j\}_{j\in\mathbb{N}}$ of $E \setminus F$ which is dense in E. Now, for each $j \in \mathbb{N}$ there exists $N_j \subseteq \Gamma_\kappa(x_j)$, null-set for μ, such that $|u(x)| \leq M$ for every $x \in \Gamma_\kappa(x_j)$. Thus, $N := \cup_{j\in\mathbb{N}} N_j \subseteq \mathscr{X} \setminus E$ is a null-set for μ and $|u(x)| \leq M$ for every point x belonging to

$$\Big(\bigcup_{j\in\mathbb{N}} \Gamma_\kappa(x_j)\Big) \setminus N = \mathcal{F}_\kappa(A) \setminus N = \mathcal{F}_\kappa(\overline{A}) \setminus N = \mathcal{F}_\kappa(E) \setminus N = (\mathscr{X} \setminus E) \setminus N, \tag{6.96}$$

where the second equality follows from (i) in Lemma 6.2, and the last equality is a consequence of (6.2) and the fact that E is a closed subset of $(\mathscr{X}, \tau_\rho)$. Hence, $\|u\|_{L^\infty(\mathscr{X}\setminus E,\mu)} \leq M$, as desired. This finishes the justification of (6.95) and finishes the proof of the theorem. $\square$

6.2 Estimates relating the Lusin and Carleson operators

We now introduce a Carleson operator $\mathfrak{C}$ and show how it can be used instead of the area operator $\mathscr{A}$ to provide an equivalent quasi-norm for the mixed norm spaces. This is essential in Subsection 6.4, and it is achieved by combining Theorem 6.8 with a good λ inequality, as in Theorem 3 of [16].

Let $(\mathscr{X}, \rho)$ be a quasi-metric space, μ a Borel measure on $(\mathscr{X}, \tau_\rho)$, E a nonempty, proper, closed subset of $(\mathscr{X}, \tau_\rho)$, and σ a measure on E such that $(E, \rho\big|_E, \sigma)$ is a space of homogeneous type. For each index $q \in (0,\infty)$ and constant $\kappa \in (0,\infty)$, recall the L^q-based Lusin (or area) operator $\mathscr{A}_{q,\kappa}$ from (6.7), and now define the L^q-based `Carleson operator` $\mathfrak{C}_{q,\kappa}$ for all μ-measurable functions $u : \mathscr{X} \setminus E \to \overline{\mathbb{R}} := [-\infty, +\infty]$ by

$$(\mathfrak{C}_{q,\kappa} u)(x) := \sup_{\Delta\subseteq E,\, x\in\Delta} \Big(\tfrac{1}{\sigma(\Delta)} \int_{\mathcal{T}_\kappa(\Delta)} |u(y)|^q \sigma(\pi^\kappa_y)\, d\mu(y)\Big)^{\frac{1}{q}}, \qquad \forall\, x \in E, \tag{6.97}$$

where π^κ_y is from (6.17), the supremum is taken over `surface balls`, i.e., sets of the form

$$\Delta := \Delta(y,r) := E \cap B_{\rho_\#}(y,r), \qquad y \in E, \quad r > 0 \tag{6.98}$$

containing x, and $\mathcal{T}_\kappa(\Delta)$ is the tent region over Δ from (6.16).

The following theorem extends the result on $\mathbb{R}^{n+1}_+$ from [16, Theorem 3, p. 318]. To state it, consider a measure space (E, σ), and for each $p \in (0,\infty)$ and $r \in (0,\infty]$, let $L^{p,r}(E,\sigma)$ denote the Lorentz space equipped with the quasi-norm

$$\|f\|_{L^{p,r}(E,\sigma)} := \Big(\int_0^\infty \lambda^r \sigma\big(\{x \in E : |f(x)| > \lambda\}\big)^{r/p} \frac{d\lambda}{\lambda}\Big)^{1/r}, \quad \text{if } r < \infty, \tag{6.99}$$

$$\|f\|_{L^{p,\infty}(E,\sigma)} := \sup_{\lambda>0} \Big[\lambda\, \sigma\left(\{x \in E : |f(x)| > \lambda\}\right)^{1/p}\Big] \quad \text{if } r = \infty. \tag{6.100}$$

Note that $L^{p,p}(E,\sigma) = L^p(E,\sigma)$ for each $p \in (0,\infty)$. Also, given a quasi-metric space $(\mathscr{X},\rho)$, call a Borel measure μ on $(\mathscr{X},\tau_\rho)$ `locally finite` when $\mu\big(B_\rho(x,r)^\circ\big) < \infty$ for all $x \in \mathscr{X}$ and $r > 0$, where the interior is taken in the topology τ_ρ.

Theorem 6.10. *Let $(\mathscr{X},\rho)$ be a quasi-metric space, μ be a locally finite Borel measure on $(\mathscr{X},\tau_\rho)$, and assume that E is a proper, nonempty, closed subset of $(\mathscr{X},\tau_\rho)$, and σ a measure on E such that $(E,\rho\big|_E,\sigma)$ is a space of homogeneous type. Fix $q \in (0,\infty)$ and $\kappa > 0$. Then the following estimates hold.*

(1) For each $p \in (0,\infty)$ there exists $C \in (0,\infty)$ such that $\|\mathscr{A}_{q,\kappa}u\|_{L^p(E,\sigma)} \leq C\|\mathfrak{C}_{q,\kappa}u\|_{L^p(E,\sigma)}$ for every μ-measurable function $u : \mathscr{X} \setminus E \to \overline{\mathbb{R}}$.

(2) For each $p \in (q,\infty)$ and each $r \in (0,\infty]$ there exists a constant $C \in (0,\infty)$ such that $\|\mathfrak{C}_{q,\kappa}u\|_{L^{p,r}(E,\sigma)} \leq C\|\mathscr{A}_{q,\kappa}u\|_{L^{p,r}(E,\sigma)}$ for every μ-measurable function $u : \mathscr{X} \setminus E \to \overline{\mathbb{R}}$.

(3) Corresponding to the end-point cases $p = q$ and $p = \infty$ in (2), there exists $C \in (0,\infty)$ such that for every μ-measurable function $u : \mathscr{X} \setminus E \to \overline{\mathbb{R}}$ the following estimates hold:

$$\|\mathfrak{C}_{q,\kappa}u\|_{L^{q,\infty}(E,\sigma)} \leq C\|\mathscr{A}_{q,\kappa}u\|_{L^q(E,\sigma)} \quad \textit{and} \quad \|\mathfrak{C}_{q,\kappa}u\|_{L^\infty(E,\sigma)} \leq C\|\mathscr{A}_{q,\kappa}u\|_{L^\infty(E,\sigma)}. \tag{6.101}$$

In particular,

$$\|\mathscr{A}_{q,\kappa}u\|_{L^p(E,\sigma)} \approx \|\mathfrak{C}_{q,\kappa}u\|_{L^p(E,\sigma)} \quad \textit{for each } p \in (q,\infty), \tag{6.102}$$

uniformly in $u : \mathscr{X} \setminus E \to \overline{\mathbb{R}}$, μ-measurable function.

Proof. Fix $q \in (0,\infty)$ and define

$$c_q := \begin{cases} 2^{(1/q)-1} & \text{if } q < 1, \\ 1 & \text{if } q \geq 1. \end{cases} \tag{6.103}$$

Also, fix an arbitrary μ-measurable function $u : \mathscr{X} \setminus E \to \overline{\mathbb{R}}$. We claim that the following good-λ inequality is valid:

$$\begin{gathered} \forall\, \kappa > 0,\ \ \exists\, \kappa' > \kappa,\ \ \exists\, c \in (0,\infty) \text{ such that } \forall\, \gamma \in (0,1],\ \forall\, \lambda \in (0,\infty), \text{ there holds} \\ \sigma\big(\{x \in E : (\mathscr{A}_{q,\kappa}u)(x) > 2c_q\lambda, (\mathfrak{C}_{q,\kappa}u)(x) \leq \gamma\lambda\}\big) \leq c\,\gamma^q \sigma\big(\{x \in E :\ (\mathscr{A}_{q,\kappa'}u)(x) > \lambda\}\big), \end{gathered} \tag{6.104}$$

where the constant $c \in (0,\infty)$ is independent of u. Suppose for now that the above claim is true. Then, if $\kappa > 0$ is fixed, let κ', c be as in (6.104). Hence, for each fixed $\gamma \in (0,1]$ and every $\lambda > 0$ we have

$$\begin{aligned} &\sigma\big(\{x \in E :\ (\mathscr{A}_{q,\kappa}u)(x) > 2c_q\lambda\}\big) \\ &\qquad \leq \sigma\big(\{x \in E :\ (\mathfrak{C}_{q,\kappa}u)(x) > \gamma\lambda\}\big) + c\,\gamma^q \sigma\big(\{x \in E :\ (\mathscr{A}_{q,\kappa'}u)(x) > \lambda\}\big). \end{aligned} \tag{6.105}$$

Thus, if we multiply the inequality in (6.105) by $p\lambda^{p-1}$ and then integrate in $\lambda \in (0,\infty)$, we obtain

$$(2c_q)^{-p}\|\mathscr{A}_{q,\kappa}u\|^p_{L^p(E,\sigma)} \leq \gamma^{-p}\|\mathfrak{C}_{q,\kappa}u\|^p_{L^p(E,\sigma)} + c\,\gamma^q\|\mathscr{A}_{q,\kappa'}u\|^p_{L^p(E,\sigma)}, \qquad \forall\, \gamma \in (0,1]. \tag{6.106}$$

Since from Theorem 6.8 we know that there exists a finite constant $C > 0$ depending only on κ, κ', p, q and geometry, such that $\|\mathscr{A}_{q,\kappa'}u\|_{L^p(E,\sigma)} \leq C\|\mathscr{A}_{q,\kappa}u\|_{L^p(E,\sigma)}$, we arrive at

$$(2c_q)^{-p}\|\mathscr{A}_{q,\kappa}u\|^p_{L^p(E,\sigma)} \leq \gamma^{-p}\|\mathfrak{C}_{q,\kappa}u\|^p_{L^p(E,\sigma)} + c\,C^p\,\gamma^q\|\mathscr{A}_{q,\kappa}u\|^p_{L^p(E,\sigma)}, \quad \forall\,\gamma \in (0,1]. \tag{6.107}$$

In order to hide the last term in the right-hand side into the left-hand side, fix a point $x_0 \in E$ and, for each $j \in \mathbb{N}$, consider

$$u_j := \min\{|u|, j\} \cdot \mathbf{1}_{B_{\rho_\#}(x_0,j)\setminus E} \qquad \text{on } \ \mathscr{X} \setminus E. \tag{6.108}$$

Observe that

$$\operatorname{supp}\big(\mathscr{A}_{q,\kappa}u_j\big) \subseteq B_{\rho_\#}\big(x_0, C_\rho(1+\kappa)j\big), \qquad 0 \leq \mathscr{A}_{q,\kappa}u_j \leq j \cdot \mu\big(B_{\rho_\#}(x_0,j)\big)^{1/q} < \infty, \tag{6.109}$$

where the last inequality uses the fact that μ is locally finite. In turn, this implies that

$$\|\mathscr{A}_{q,\kappa}u_j\|_{L^p(E,\sigma)} \leq j \cdot \mu\big(B_{\rho_\#}(x_0,j)\big)^{1/q}\sigma\big(E \cap B_{\rho_\#}(x_0, C_\rho(1+\kappa)j)\big)^{1/p} < \infty, \tag{6.110}$$

since σ is locally finite. Hence, if we choose $\gamma \in (0,1]$ so that $(2c_q)^{-p} > 2cC^p\,\gamma^q$, then we obtain from (6.107) written with u replaced by u_j

$$\|\mathscr{A}_{q,\kappa}u_j\|^p_{L^p(E,\sigma)} \leq C\|\mathfrak{C}_{q,\kappa}u_j\|^p_{L^p(E,\sigma)}, \qquad \forall\,j \in \mathbb{N}, \tag{6.111}$$

for some $C \in (0,\infty)$ independent of j. Note that $0 \leq \mathfrak{C}_{q,\kappa}u_j \leq \mathfrak{C}_{q,\kappa}u$ pointwise in E and that $u_j \nearrow |u|$ pointwise μ-a.e. on $\mathscr{X} \setminus E$ implies $\mathscr{A}_{q,\kappa}u_j \nearrow \mathscr{A}_{q,\kappa}u$ everywhere on E by Lebesgue's Monotone Convergence Theorem. Based on these observations, (6.111) and Fatou's lemma we may then conclude that

$$\begin{aligned}\|\mathscr{A}_{q,\kappa}u\|^p_{L^p(E,\sigma)} &= \int_E \liminf_{j\to\infty}\big[\mathscr{A}_{q,\kappa}u_j\big]^p\,d\sigma \leq \liminf_{j\to\infty}\int_E\big[\mathscr{A}_{q,\kappa}u_j\big]^p\,d\sigma \\ &= \liminf_{j\to\infty}\|\mathscr{A}_{q,\kappa}u_j\|^p_{L^p(E,\sigma)} \leq C\|\mathfrak{C}_{q,\kappa}u\|^p_{L^p(E,\sigma)}.\end{aligned} \tag{6.112}$$

That is, granted (6.104), we have

$$\|\mathscr{A}_{q,\kappa}u\|^p_{L^p(E,\sigma)} \leq C\|\mathfrak{C}_{q,\kappa}u\|^p_{L^p(E,\sigma)}. \tag{6.113}$$

Thus, to finish the proof of part *(1)* of the statement of the theorem, we are left with proving (6.104). Fix $\kappa' > \kappa > 0$ along with $\gamma \in (0,1]$, then for an arbitrary $\lambda > 0$ define the set

$$\mathcal{O}_\lambda := \big\{x \in E : \,(\mathscr{A}_{q,\kappa'}u)(x) > \lambda\big\}. \tag{6.114}$$

By Lemma 6.1, $\mathcal{O}_\lambda$ is an open subset of $(E, \tau_{\rho|_E})$. Also, since $\mathscr{A}_{q,\kappa'}u \geq \mathscr{A}_{q,\kappa}u$ pointwise in E, we conclude that

$$\{x \in E : \,(\mathscr{A}_{q,\kappa}u)(x) > 2c_q\lambda\} \subseteq \mathcal{O}_\lambda. \tag{6.115}$$

If $\mathcal{O}_\lambda = \emptyset$, then by (6.115) the inequality in the second line of (6.104) is trivially satisfied. Therefore, assume that $\mathcal{O}_\lambda \neq \emptyset$ in what follows. Let us also make the assumption (which will be eliminated *a posteriori*) that

$$\begin{gathered}\text{the } \mu\text{-measurable function } u : \mathscr{X} \setminus E \to \overline{\mathbb{R}} \text{ is such that} \\ \mathcal{O}_\lambda \text{ from (6.114) is a proper subset of } E \text{ for each } \lambda > 0.\end{gathered} \tag{6.116}$$

In such a scenario, for a fixed, suitably chosen $\lambda_o > 1$ we may apply Proposition 2.6 (with λ there replaced by λ_o) to obtain a Whitney covering of $\mathcal{O}_\lambda$ by balls, relative to $(E, \rho|_E)$, which we may assume (given the freedom of choosing the parameter λ_o, and (2.15)) to be of the form $B_j := E \cap B_{\rho_\#}(x_j, r_j)$, $j \in \mathbb{N}$, satisfying properties *(1)-(4)* in Proposition 2.6 for some $\Lambda > \lambda_o$. If we now prove that

$$
\begin{gathered}
\exists\, \kappa' > \kappa \text{ and } \exists\, c \in (0,\infty) \text{ such that } \forall\, \gamma \in (0,1],\ \forall\, \lambda \in (0,\infty), \text{ there holds} \\
\sigma\big(\{x \in B_j : (\mathscr{A}_{q,\kappa} u)(x) > 2c_q\lambda,\ (\mathfrak{C}_{q,\kappa} u)(x) \leq \gamma\lambda\}\big) \leq c\,\gamma^q \sigma(B_j) \ \text{ for every } \ j \in \mathbb{N},
\end{gathered}
\tag{6.117}
$$

then combining (6.117) with (6.115) and properties *(1)-(2)* from Proposition 2.6, we may estimate

$$
\begin{aligned}
\sigma\big(\{x \in E &: (\mathscr{A}_{q,\kappa} u)(x) > 2c_q\lambda, (\mathfrak{C}_{q,\kappa} u)(x) \leq \gamma\lambda\}\big) \\
&= \sigma\big(\{x \in \mathcal{O}_\lambda :\ (\mathscr{A}_{q,\kappa} u)(x) > 2c_q\lambda,\ (\mathfrak{C}_{q,\kappa} u)(x) \leq \gamma\lambda\}\big) \\
&\leq \sum_{j=1}^{\infty} \sigma\big(\{x \in B_j : (\mathscr{A}_{q,\kappa} u)(x) > 2c_q\lambda, (\mathfrak{C}_{q,\kappa} u)(x) \leq \gamma\lambda\}\big) \leq c\,\gamma^q \sum_{j=1}^{\infty} \sigma(B_j) \\
&\leq C\gamma^q \sigma(\mathcal{O}_\lambda).
\end{aligned}
\tag{6.118}
$$

Hence, (6.104) follows.

Turning now to the proof of (6.117), fix $j \in \mathbb{N}$, and note that without loss of generality we may assume that

$$
\big\{x \in B_j : (\mathscr{A}_{q,\kappa} u)(x) > 2c_q\lambda,\ (\mathfrak{C}_{q,\kappa} u)(x) \leq \gamma\lambda\big\} \neq \emptyset,
\tag{6.119}
$$

since otherwise there is nothing to prove. Decompose $u = u\mathbf{1}_{\{\delta_E \geq r_j\}} + u\mathbf{1}_{\{\delta_E < r_j\}} =: u_1 + u_2$ and let $z_j \in E \setminus \mathcal{O}_\lambda$ be such that $\rho_\#(x_j, z_j) \leq \Lambda r_j$ (the existence of z_j is guaranteed by property *(3)* in Proposition 2.6). We claim that

$$
\begin{gathered}
\text{there exists } \ \kappa' > \kappa \ \text{ independent of } j \in \mathbb{N} \text{ with the property that} \\
\text{if } \ x \in B_j \ \text{ and } \ y \in \Gamma_\kappa(x) \ \text{ is such that } \ \delta_E(y) \geq r_j \ \text{ then } \ y \in \Gamma_{\kappa'}(z_j).
\end{gathered}
\tag{6.120}
$$

Indeed, if $x \in B_j$, we have $\rho_\#(x, z_j) \leq C_{\rho_\#} \max\{\rho_\#(x, x_j), \rho_\#(x_j, z_j)\} \leq C_\rho \Lambda r_j$. Hence, if $y \in \Gamma_\kappa(x)$ is such that $\delta_E(y) \geq r_j$ then

$$
\begin{aligned}
\rho_\#(y, z_j) &\leq C_{\rho_\#} \max\{\rho_\#(y, x), \rho_\#(x, z_j)\} \leq C_\rho \max\{(1+\kappa)\delta_E(y), C_\rho \Lambda r_j\} \\
&\leq C_\rho \max\{(1+\kappa), C_\rho \Lambda\} \delta_E(y).
\end{aligned}
\tag{6.121}
$$

Now we choose $\kappa' > C_\rho \max\{(1+\kappa), C_\rho\Lambda\} - 1$, so then $\kappa' > \kappa$, and κ' depends only on finite positive geometrical constants (hence, in particular, it is independent of $j \in \mathbb{N}$). Based on (6.121) we obtain that (6.120) holds true for this choice of κ'. Then, using (6.120) and recalling that $z_j \in E \setminus \mathcal{O}_\lambda$, we may write

$$
\begin{aligned}
(\mathscr{A}_{q,\kappa} u_1)(x)^q &= \int_{\substack{y \in \Gamma_\kappa(x) \\ \delta_E(y) \geq r_j}} |u(y)|^q \, d\mu(y) \leq \int_{y \in \Gamma_{\kappa'}(z_j)} |u(y)|^q \, d\mu(y) \\
&= (\mathscr{A}_{q,\kappa'} u)(z_j)^q \leq \lambda^q, \qquad \forall\, x \in B_j.
\end{aligned}
\tag{6.122}
$$

Next, we make use of (6.58) to write

$$\begin{aligned}\int_{B_j} (\mathscr{A}_{q,\kappa} u_2)(x)^q \, d\sigma(x) &= \int\limits_{y\in\mathcal{F}_\kappa(B_j),\, \delta_E(y)<r_j} |u(y)|^q \sigma\big(B_j \cap \pi_y^\kappa\big)\, d\mu(y) \\ &\leq \int\limits_{y\in\mathcal{F}_\kappa(B_j),\, \delta_E(y)<r_j} |u(y)|^q \sigma(\pi_y^\kappa)\, d\mu(y). \end{aligned} \tag{6.123}$$

In order to proceed further, first make a geometrical observation to the effect that (using notation introduced in (6.98))

$$\begin{array}{c}\text{there exists a finite constant } c_o > 0 \text{ such that for every } r > 0 \text{ and every } x_0 \in E \\ \text{if } \ y \in \mathcal{F}_\kappa(\Delta(x_0, r)) \ \text{ and } \ \delta_E(y) < r \ \text{ then } \ y \in \mathcal{T}_\kappa\big(E \cap B_{\rho_\#}(w, c_o r)\big) \ \forall\, w \in \Delta(x_0, r).\end{array} \tag{6.124}$$

To see why this is true, consider a point $y \in \mathcal{F}_\kappa(\Delta(x_0, r))$ with the property that $\delta_E(y) < r$. Then there exists $x \in \Delta(x_0, r)$ such that $\rho_\#(y, x) < (1+\kappa)\delta_E(y) < (1+\kappa)r$. Let $w \in \Delta(x_0, r)$ be arbitrary and note that

$$\rho_\#(x, w) \leq C_{\rho_\#} \max\{\rho_\#(x, x_0), \rho_\#(x_0, w)\} < C_\rho r. \tag{6.125}$$

Accordingly, choosing $c_o > C_\rho$ forces $x \in E \cap B_{\rho_\#}(w, c_o r)$ hence, further,

$$\mathrm{dist}_{\rho_\#}\big(y, E \cap B_{\rho_\#}(w, c_o r)\big) \leq \rho_\#(y, x) < (1+\kappa)\delta_E(y). \tag{6.126}$$

Let us also observe that

$$\begin{aligned}\rho_\#(w, y) &\leq C_{\rho_\#} \max\{\rho_\#(w, x), \rho_\#(x, y)\} \\ &\leq C_{\rho_\#} \max\Big\{C_{\rho_\#} \max\{\rho_\#(w, x_0), \rho_\#(x_0, x)\},\ \rho_\#(x, y)\Big\} \\ &\leq C_\rho \max\{C_\rho, 1+\kappa\} r. \end{aligned} \tag{6.127}$$

Thus, if $z \in E \setminus B_{\rho_\#}(w, c_o r)$, making use of (6.127) we obtain

$$\begin{aligned} c_o r &\leq \rho_\#(z, w) \leq C_{\rho_\#} \max\{\rho_\#(z, y), \rho_\#(y, w)\} \\ &\leq C_\rho \max\Big\{\rho_\#(z, y),\ C_\rho \max\{C_\rho, 1+\kappa\} r\Big\} = C_\rho\, \rho_\#(z, y), \end{aligned} \tag{6.128}$$

where the last equality is necessarily true if we take $c_o > C_\rho^2 \max\{C_\rho, 1+\kappa\}$ (given the nature of the left-most side of (6.128)). Consequently, for this choice of c_o, (6.128) gives that

$$\rho_\#(y, z) \geq \frac{c_o}{C_\rho} r > \frac{c_o}{C_\rho} \delta_E(y), \qquad \forall\, z \in E \setminus B_{\rho_\#}(w, c_o r) \tag{6.129}$$

hence, if we also assume $c_o \geq C_\rho(1+\kappa)^2$, then

$$\mathrm{dist}_{\rho_\#}\big(y, E \setminus B_{\rho_\#}(w, c_o r)\big) \geq (1+\kappa)^2 \delta_E(y). \tag{6.130}$$

Together, (6.126) and (6.130) allow us to conclude that if $c_o > \max\big\{C_\rho(1+\kappa)^2, C_\rho^3, C_\rho^2(1+\kappa)\big\}$ then

$$\mathrm{dist}_{\rho_\#}\big(y, E \cap B_{\rho_\#}(w, c_o r)\big) \leq (1+\kappa)^{-1} \mathrm{dist}_{\rho_\#}\big(y, E \setminus B_{\rho_\#}(w, c_o r)\big). \tag{6.131}$$

In light of (6.18), we deduce from (6.131) that $y \in \mathcal{T}_\kappa\big(E \cap B_{\rho_\#}(w, c_o r)\big)$ when c_o is chosen as above. This completes the proof of (6.124).

Combining (6.123) with (6.124) (the latter applied with B_j in place of $\Delta(x_0, r)$), and keeping in mind that

$$\sigma(B_j) \approx \sigma\big(E \cap B_{\rho_\#}(w, c_o r_j)\big), \text{ uniformly in } j \in \mathbb{N} \text{ and } w \in B_j, \tag{6.132}$$

which is a consequence of (2.37), we may then estimate

$$\begin{aligned} \tfrac{1}{\sigma(B_j)} \int_{B_j} (\mathscr{A}_{q,\kappa} u_2)(x)^q \, d\sigma(x) &\leq \tfrac{C}{\sigma(B_j)} \int\limits_{y \in \mathcal{F}_\kappa(B_j),\, \delta_E(y) < r_j} |u(y)|^q \sigma(\pi_y^\kappa) \, d\mu(y) \\ &\leq \frac{C}{\sigma\Big(E \cap B_{\rho_\#}(w, c_o r_j)\Big)} \int_{\mathcal{T}_\kappa\Big(E \cap B_{\rho_\#}(w, c_o r_j)\Big)} |u(y)|^q \sigma(\pi_y^\kappa) \, d\mu(y) \\ &\leq C \inf_{w \in B_j} \big[(\mathfrak{C}_{q,\kappa} u)(w)\big]^q \leq C \gamma^q \lambda^q, \end{aligned} \tag{6.133}$$

where for the last inequality in (6.133) we have used the assumption (6.119). In concert with Tschebyshev's inequality, (6.133) gives that

$$\sigma\big(\{x \in B_j : (\mathscr{A}_{q,\kappa} u_2)(x) > \lambda\}\big) \leq C \gamma^q \sigma(B_j), \tag{6.134}$$

for some $C \in (0, \infty)$ independent of $\gamma \in (0, 1]$ and $j \in \mathbb{N}$. Also, in view of (6.122), we obtain

$$\big\{x \in B_j : (\mathscr{A}_{q,\kappa} u)(x) > 2c_q \lambda\big\} \subseteq \big\{x \in B_j : (\mathscr{A}_{q,\kappa} u_2)(x) > \lambda\big\}, \tag{6.135}$$

since pointwise on E we have $\mathscr{A}_{q,\kappa} u \leq c_q\big(\mathscr{A}_{q,\kappa} u_1 + \mathscr{A}_{q,\kappa} u_2\big)$, where c_q is as in (6.103). Combined with (6.134), this yields the inequality in (6.117). The proof of part *(1)* of the theorem is then complete, provided we dispense with the additional hypothesis in (6.116). To do this, we distinguish two cases.

`Case I:` *Assume that* $\operatorname{diam}_\rho(E) = \infty$. An inspection of the proof reveals that estimate (6.107) has only been utilized with u_j (from (6.108)) in place of u. As such, we only need to know that $\big\{x \in E : (\mathscr{A}_{q,\kappa'} u_j)(x) > \lambda\big\}$ is a proper subset of E for each $j \in \mathbb{N}$ and each $\lambda > 0$. However, in the case we are currently considering, this follows by observing that, on the one hand, $\sigma(E) = \infty$ by (2.38), while on the other hand $\sigma\big(\big\{x \in E : (\mathscr{A}_{q,\kappa'} u_j)(x) > \lambda\big\}\big) < \infty$ by (6.110) and Tschebyshev's inequality.

`Case II:` *Assume that* $\operatorname{diam}_\rho(E) < \infty$. Recall from (2.38) that this entails $\sigma(E) < \infty$, and set $R := \operatorname{diam}_{\rho_\#}(E) \in (0, \infty)$. For some positive, small number ε_o, to be specified later, decompose $|u| = u' + u'' := |u|\mathbf{1}_{\{\delta_E(\cdot) < \varepsilon_o R\}} + |u|\mathbf{1}_{\{\delta_E(\cdot) \geq \varepsilon_o R\}}$. Hence, u', u'' are μ-measurable and $0 \leq u', u'' \leq |u|$. Note that for each $x \in E$, (6.97) gives

$$\begin{aligned} (\mathfrak{C}_{q,\kappa} u'')(x) &\geq \Big(\tfrac{1}{\sigma(E)} \int_{\mathscr{X} \setminus E} u''(y)^q \sigma(\pi_y^\kappa) \, d\mu(y)\Big)^{1/q} \\ &\geq c\Big(\int\limits_{\substack{y \in \mathscr{X} \setminus E \\ \delta_E(y) \geq \varepsilon_o R}} u''(y)^q \, d\mu(y)\Big)^{1/q} \geq c(\mathscr{A}_{q,\kappa} u'')(x), \end{aligned} \tag{6.136}$$

by taking $r > R$ in (6.98) and recalling (iv) in Lemma 6.2, and observing that there exists a constant $C \in (0,\infty)$ with the property that for each $y \in \mathscr{X} \setminus E$ we have (with y_* and ϵ as in Lemma 6.3)

$$\sigma(\pi_y^\kappa) \geq \sigma\big(E \cap B_{\rho_\#}(y_*, \epsilon\delta_E(y))\big) \geq \sigma\big(E \cap B_{\rho_\#}(y_*, \epsilon\varepsilon_o R)\big) \geq C\sigma(E), \tag{6.137}$$

where the last inequality is a consequence of the doubling condition on σ. In turn, (6.136) and the monotonicity of the Carleson operator allow us to write

$$\|\mathscr{A}_{q,\kappa}u''\|_{L^p(E,\sigma)} \leq C\|\mathfrak{C}_{q,\kappa}u''\|_{L^p(E,\sigma)} \leq C\|\mathfrak{C}_{q,\kappa}u\|_{L^p(E,\sigma)}. \tag{6.138}$$

To proceed, set $\varepsilon_o := \frac{1}{4C_\rho(1+\kappa')}$ and fix $x_1, x_2 \in E$ satisfying $\rho_\#(x_1,x_2) > R/2$. We claim that these choices guarantee that

$$\Gamma_{\kappa'}(x_1) \cap \Gamma_{\kappa'}(x_2) \subseteq \{x \in \mathscr{X} \setminus E :\, \delta_E(x) > \varepsilon_0 R\}. \tag{6.139}$$

Indeed, if $y \in \Gamma_{\kappa'}(x_1) \cap \Gamma_{\kappa'}(x_2)$ then $\rho_\#(y,x_j) < (1+\kappa')\delta_E(y)$ for $j = 1,2$ and we have $R/2 < \rho_\#(x_1,x_2) \leq C_\rho \max\{\rho_\#(y,x_1), \rho_\#(y,x_2)\} < C_\rho(1+\kappa')\delta_E(y) = \frac{1}{4\varepsilon_o}\delta_E(y)$, which shows that the inclusion in (6.139) is true. If we now further decompose

$$u' = u_1' + u_2' := u'\mathbf{1}_{\Gamma_{\kappa'}(x_1)} + u'\big(1 - \mathbf{1}_{\Gamma_{\kappa'}(x_1)}\big) \tag{6.140}$$

then $0 \leq u_1', u_2' \leq u'$, and both u_1', u_2' are μ-measurable. Moreover, due to (6.139) and the fact that u_1 has support contained in the set $\{\delta_E(\cdot) < \varepsilon_o R\}$, we also obtain that $(\mathscr{A}_{q,\kappa'}u_1')(x_2) = 0$ and $(\mathscr{A}_{q,\kappa'}u_2')(x_1) = 0$. The latter imply that the sets constructed according to the same recipe as $\mathcal{O}_\lambda$ in (6.114) but with u replaced by either u_1' or u_2', are proper subsets of E for every $\lambda > 0$. Hence, hypothesis (6.116) holds for each of the functions u_1', u_2'. As such, the first part of the proof gives that (6.113) holds with u replaced by either u_1' or u_2'. In concert, these give

$$\begin{aligned} \|\mathscr{A}_{q,\kappa}u'\|_{L^p(E,\sigma)} &\leq C\|\mathscr{A}_{q,\kappa}u_1'\|_{L^p(E,\sigma)} + C\|\mathscr{A}_{q,\kappa}u_2'\|_{L^p(E,\sigma)} \\ &\leq C\|\mathfrak{C}_{q,\kappa}u_1'\|_{L^p(E,\sigma)} + C\|\mathfrak{C}_{q,\kappa}u_2'\|_{L^p(E,\sigma)} \leq C\|\mathfrak{C}_{q,\kappa}u\|_{L^p(E,\sigma)}. \end{aligned} \tag{6.141}$$

Together with (6.138), this then yields (6.113) for the original function u. This finishes the treatment of Case II and completes the proof of the estimate in part *(1)* of the theorem.

Moving on to the proof of part *(2)*, the key step is establishing the pointwise estimate

$$(\mathfrak{C}_{q,\kappa}u)(x_0) \leq C\big[M_E(\mathscr{A}_{q,\kappa}u)^q(x_0)\big]^{\frac{1}{q}}, \qquad \forall\, x_0 \in E, \tag{6.142}$$

for some $C \in (0,\infty)$ depending only on κ, p, q and geometrical characteristics of the ambient space. To justify this, fix $r > 0$, and let Δ be a ball of radius r in $(E, (\rho|_E)_\#)$. Then, upon recalling (6.58), (ii) in Lemma 6.2, and (6.19), we may write

$$\begin{aligned} \int_\Delta (\mathscr{A}_{q,\kappa}u)(x)^q\, d\sigma(x) &= \int_{\mathcal{F}_\kappa(\Delta)} |u(y)|^q \sigma\big(\Delta \cap \pi_y^\kappa\big)\, d\mu(y) \\ &\geq \int_{\mathcal{T}_\kappa(\Delta)} |u(y)|^q \sigma\big(\Delta \cap \pi_y^\kappa\big)\, d\mu(y) \\ &= \int_{\mathcal{T}_\kappa(\Delta)} |u(y)|^q \sigma\big(\pi_y^\kappa\big)\, d\mu(y). \end{aligned} \tag{6.143}$$

Now (6.142) follows from (6.143) by dividing the latter inequality by $\sigma(\Delta)$ and taking the supremum over all Δ's containing an arbitrary given point $x_0 \in E$.

With the pointwise estimate (6.142) in hand, whenever $p \in (q, \infty)$ and $r \in (0, \infty]$ we may use the boundedness of M_E on the Lorentz space $L^{p/q, r/q}(E, \sigma)$, which holds since $p/q > 1$, and the general fact that for each $\alpha > 0$ we have

$$\| |f|^\alpha \|_{L^{p,r}(E,\sigma)} = C(p, r, \alpha) \|f\|^\alpha_{L^{p\alpha, r\alpha}(E,\sigma)}, \tag{6.144}$$

in order to write

$$\begin{aligned} \|\mathfrak{C}_{q,\kappa} u\|_{L^{p,r}(E,\sigma)} &\leq C \big\| [M_E(\mathscr{A}_{q,\kappa} u)^q]^{\frac{1}{q}} \big\|_{L^{p,r}(E,\sigma)} = C \|M_E(\mathscr{A}_{q,\kappa} u)^q\|^{\frac{1}{q}}_{L^{p/q,r/q}(E,\sigma)} \\ &\leq C \|(\mathscr{A}_{q,\kappa} u)^q\|^{\frac{1}{q}}_{L^{p/q,r/q}(E,\sigma)} = C \|\mathscr{A}_{q,\kappa} u\|_{L^{p,r}(E,\sigma)}, \end{aligned} \tag{6.145}$$

as required.

There remains to observe that the two estimates in *(3)* are obtained by a computation similar to (6.145) that is based on (6.142), the weak-$(1,1)$ boundedness of M_E, and the boundedness of M_E on $L^\infty(E, \sigma)$. This finishes the proof of the theorem. $\square$

Remark 6.11. *The case $p = q = r$ of part (2) of Theorem 6.10, which corresponds to the estimate $\|\mathfrak{C}_{p,\kappa} u\|_{L^p(E,\sigma)} \leq C \|\mathscr{A}_{p,\kappa} u\|_{L^p(E,\sigma)}$, fails in general. A counterexample in Euclidean space when $p = 2$ is given in the remarks stated below Theorem 3 of [16].*

6.3 Weak L^p square function estimates imply L^2 square function estimates

We are now in a position to consider L^p versions of the L^2 square function estimates considered in Section 3 for integral operators Θ_E. The main result is that L^2 square function estimates follow automatically from weak L^p square function estimates for any $p \in (0, \infty)$. This is stated in Theorem 6.12 below. The result is achieved by combining the $T(1)$ theorem in Theorem 3.2 with a weak type John-Nirenberg lemma for Carleson measures based on Lemma 2.14 in [4] (see also [26, Lemma IV.1.12] for a similar result).

Theorem 6.12. *Let $0 < d < m < \infty$. Assume that $(\mathscr{X}, \rho, \mu)$ is an m-dimensional* ADR *space, E is a closed subset of $(\mathscr{X}, \tau_\rho)$, and σ is a Borel regular measure on $(E, \tau_{\rho|_E})$ with the property that $(E, \rho\big|_E, \sigma)$ is a d-dimensional* ADR *space. Finally, suppose that Θ is the integral operator defined in* (3.4) *with a kernel θ as in* (3.1), (3.2), (3.3).

Then whenever $\kappa, p, C_o \in (0, \infty)$ are such that for every surface ball $\Delta \subseteq E$ (cf. (6.98)*)*

$$\sigma\left(\left\{x \in E : \int_{\Gamma_\kappa(x)} |(\Theta \mathbf{1}_\Delta)(y)|^2 \, \delta_E(y)^{2\upsilon - m} \, d\mu(y) > \lambda^2 \right\}\right) \leq C_o \lambda^{-p} \sigma(\Delta), \quad \forall \, \lambda > 0, \tag{6.146}$$

there exists some $C \in (0, \infty)$ which depends only on κ, p, C_o and finite positive background constants (including $\operatorname{diam}_\rho(E)$ *in the case when E is bounded) with the property that*

$$\int_{\mathscr{X} \setminus E} |(\Theta f)(x)|^2 \delta_E(x)^{2\upsilon - (m-d)} \, d\mu(x) \leq C \int_E |f(x)|^2 \, d\sigma(x), \qquad \forall \, f \in L^2(E, \sigma). \tag{6.147}$$

The requirement in (6.146) is actually less restrictive than a weak L^p square function estimate. In particular, it is satisfied whenever the following weak L^p square function estimate holds for every $f \in L^p(E,\sigma)$:

$$\sup_{\lambda>0}\left[\lambda\cdot\sigma\Big(\Big\{x\in E:\int_{\Gamma_\kappa(x)}|(\Theta f)(y)|^2\delta_E(y)^{2\upsilon-m}\,d\mu(y)>\lambda^2\Big\}\Big)^{1/p}\right]\le C_o\|f\|_{L^p(E,\sigma)}. \tag{6.148}$$

Indeed, (6.146) follows by specializing (6.148) to the case when $f=\mathbf{1}_\Delta$ for an arbitrary surface ball $\Delta\subseteq E$.

To prove Theorem 6.12, we need only set $q=2$ in Proposition 6.13 below to obtain a Carleson measure estimate, and then apply the $T(1)$ theorem for square functions in Theorem 3.2. Therefore, the remainder of this subsection is dedicated to the proof of the following proposition.

Proposition 6.13. *Retain the same background hypotheses as in the statement of Theorem 6.12. In this context, let $\mathbb{D}(E)$ denote a dyadic cube structure on E, consider a Whitney covering $\mathbb{W}_\lambda(\mathscr{X}\setminus E)$ of $\mathscr{X}\setminus E$ as in Lemma 2.21 and, corresponding to these, recall the dyadic Carleson tents from* (2.131)*. Then whenever $\kappa,p,q,C_o\in(0,\infty)$ are such that for every surface ball $\Delta\subseteq E$ there holds*

$$\sigma\left(\Big\{x\in E:\int_{\Gamma_\kappa(x)}|(\Theta\mathbf{1}_\Delta)(y)|^q\,\delta_E(y)^{q\upsilon-m}\,d\mu(y)>\lambda^q\Big\}\right)\le C_o\lambda^{-p}\sigma(\Delta),\quad\forall\,\lambda>0, \tag{6.149}$$

there exists some $C\in(0,\infty)$ which depends only on κ,p,q,C_o and finite positive background constants with the property that

$$\sup_{Q\in\mathbb{D}(E)}\Big(\tfrac{1}{\sigma(Q)}\int_{T_E(Q)}|(\Theta 1)(x)|^q\delta_E(x)^{q\upsilon-(m-d)}\,d\mu(x)\Big)\le C. \tag{6.150}$$

In preparation for presenting the proof of Proposition 6.13 we discuss a couple of auxiliary results. The first such result is a variation on the theme of Whitney decomposition discussed in Proposition 2.6.

Lemma 6.14. *Let (E,ρ,σ) be a space of homogeneous type with the property that the measure σ is Borel regular, and let $\mathbb{D}(E)$ be a collection of dyadic cubes as in Proposition 2.12. Also, suppose that $\mathcal{O}$ is an open subset of (E,τ_ρ) with the property that $\big(\mathcal{O},\rho\big|_{\mathcal{O}},\sigma\lfloor\mathcal{O}\big)$ is a space of homogeneous type. Fix $\lambda\in(1,\infty)$ and suppose Ω is an open, proper, non-empty subset of $\mathcal{O}$. Then there exist $\varepsilon\in(0,1)$, $N\in\mathbb{N}$, $\Lambda\in(\lambda,\infty)$ and a subset $\mathcal{W}\subseteq\mathbb{D}(E)$ such that the following properties are satisfied:*

(1) $Q\subseteq\Omega$ for every $Q\in\mathcal{W}$ and $\sigma\big(\Omega\setminus\bigcup_{Q\in\mathcal{W}}Q\big)=0$;

(2) $Q\cap Q'=\emptyset$ for every $Q,Q'\in\mathcal{W}$ with $Q\neq Q'$;

(3) for every $x\in\Omega$, the cardinality of the set $\big\{Q\in\mathcal{W}:\ B_\rho\big(x,\varepsilon\,\mathrm{dist}_\rho(x,\mathcal{O}\setminus\Omega)\big)\cap Q\neq\emptyset\big\}$ is at most N;

(4) $\lambda Q\subseteq\Omega$ and $\Lambda Q\cap[\mathcal{O}\setminus\Omega]\neq\emptyset$ for every $Q\in\mathcal{W}$;

(5) $\ell(Q)\approx\ell(Q')$ uniformly for $Q,Q'\in\mathcal{W}$ such that $\lambda Q\cap\lambda Q'\neq\emptyset$;

(6) $\sum\limits_{Q\in\mathcal{W}} \mathbf{1}_{\lambda Q} \leq N$.

Proof. Given $\lambda \in (1,\infty)$, apply Proposition 2.6 to the open, proper, non-empty subset Ω of the space of homogeneous type $\big(\mathcal{O}, \rho\big|_{\mathcal{O}}, \sigma\lfloor\mathcal{O}\big)$. This guarantees the existence of parameters $\varepsilon \in (0,1)$, $N \in \mathbb{N}$, $\Lambda \in (\lambda,\infty)$, as well as a covering of Ω with balls $\Omega = \bigcup_{j\in\mathbb{N}}\big(\mathcal{O}\cap B_\rho(x_j,r_j)\big)$ such that the analogues of the properties *(1)-(4)* in Proposition 2.6 hold in the current setting. Next, for each $j \in \mathbb{N}$ consider

$$I_j := \big\{Q \in \mathbb{D}(E) : \ell(Q) \approx r_j \text{ and } Q \cap B_\rho(x_j,r_j) \neq \emptyset\big\}, \tag{6.151}$$

and define $\mathcal{W} := \bigcup_{j\in\mathbb{N}} I_j$ thinned out, so that $Q\cap Q' = \emptyset$ for every $Q, Q' \in \mathcal{W}$, $Q \neq Q'$. Granted the properties the families $\mathbb{D}(E)$ and $\{B_\rho(x_j,r_j)\}_{j\in\mathbb{N}}$ satisfy (as listed in Proposition 2.12 and Proposition 2.6) and given the nature of the construction of the family $\mathcal{W}$, it follows that properties *(1)-(6)* in the statement of the current lemma hold for the family $\mathcal{W}$. $\square$

We now state the aforementioned weak type John-Nirenberg lemma for Carleson measures, cf. [4, Lemma 2.14] for a result similar in spirit in the Euclidean setting.

Lemma 6.15. *Retain the same background hypotheses as in the statement of Theorem 6.12. In this context, fix two finite numbers $\kappa, \eta > 0$, an index $q \in (0,\infty)$ and, for each $Q \in \mathbb{D}(E)$, define*

$$S_Q(x) := \Big(\int\limits_{\substack{y\in\Gamma_\kappa(x)\\ \rho_\#(x,y)<\eta\ell(Q)}} |(\Theta 1)(y)|^q \delta_E(y)^{qv-m}\, d\mu(y)\Big)^{\frac{1}{q}}, \qquad \forall\, x \in E. \tag{6.152}$$

Assuming that η is sufficiently large (depending only on geometry) and granted that there exist two parameters $N \in (0,\infty)$ and $\beta \in (0,1)$ such that

$$\sigma\Big(\big\{x \in Q :\, S_Q(x) > N\big\}\Big) < (1-\beta)\sigma(Q), \qquad \forall\, Q \in \mathbb{D}(E), \tag{6.153}$$

then one may find $C \in (0,\infty)$ depending only on geometry, the estimates satisfied by the kernel θ, and κ, η, with the property that

$$\sup_{Q\in\mathbb{D}(E)} \Big(\tfrac{1}{\sigma(Q)} \int_{T_E(Q)} |(\Theta 1)(x)|^q \delta_E(x)^{qv-(m-d)}\, d\mu(x)\Big) \leq \beta^{-1}(C + N^q). \tag{6.154}$$

Proof. For each $i \in \mathbb{N}$, let Θ_i be as in (3.94) and associate to Θ_i the function S^i_Q, much as S_Q is associated to Θ. Note that S^i_Q and S_Q depend on the constant κ defining Γ_κ. We fix $\widetilde{\kappa} \in (0,\kappa)$ to be specified later and we use the notation $S^i_{Q,\widetilde{\kappa}}$ for the function defined similarly to S^i_Q but with $\widetilde{\kappa}$ in place of κ. Also, with $N \in (0,\infty)$ and $\beta \in (0,1)$ satisfying (6.153), define

$$\Omega^{N,i}_Q := \big\{x \in Q :\, S^i_Q(x) > N\big\}, \qquad \forall\, Q \in \mathbb{D}(E), \quad \forall\, i \in \mathbb{N}. \tag{6.155}$$

Since, thanks to Lemma 6.1, for each $Q \in \mathbb{D}(E)$ and $i \in \mathbb{N}$ the function S^i_Q is lower semi-continuous, from (6.155), (6.153) and the fact that $S^i_Q \leq S_Q$ pointwise in Q we deduce that

$$\forall\, Q \in \mathbb{D}(E), \quad \forall\, i \in \mathbb{N}, \quad \Omega^{N,i}_Q \text{ is an open, proper subset of } Q. \tag{6.156}$$

To proceed, consider

$$A^i := \sup_{Q\in\mathbb{D}(E)} \Big(\tfrac{1}{\sigma(Q)} \int_Q (S^i_{Q,\widetilde{\kappa}}(x))^q \, d\sigma(x)\Big), \qquad \forall\, i \in \mathbb{N}. \tag{6.157}$$

Then, based on (6.58) (applied to the function $u := |(\Theta_i 1)|^q \delta_E^{qv-m} \mathbf{1}_{\{\mathrm{dist}_{\rho_\#}(\cdot,Q)\leq C\ell(Q)\}}$) we may write, with x_Q denoting the center of $Q \in \mathbb{D}(E)$,

$$\begin{aligned}
\int_Q (S^i_{Q,\widetilde{\kappa}}(x))^q \, d\sigma(x) \;&\leq\; \int_Q \Big(\int\limits_{\substack{y\in\Gamma_{\widetilde{\kappa}}(x) \\ \mathrm{dist}_{\rho_\#}(y,Q)\leq C\ell(Q)}} |(\Theta_i 1)(y)|^q \delta_E(y)^{qv-m} \, d\mu(y)\Big) \, d\sigma(x) \\
&\leq \int\limits_{\substack{y\in\mathcal{F}_{\widetilde{\kappa}}(Q) \\ \mathrm{dist}_{\rho_\#}(y,Q)\leq C\ell(Q)}} |(\Theta_i 1)(y)|^q \delta_E(y)^{qv-m} \sigma\big(Q \cap \pi_y^{\widetilde{\kappa}}\big) \, d\mu(y) \\
&\leq C \int\limits_{B_{\rho_\#}(x_Q, C\ell(Q))} |(\Theta_i 1)(y)|^q \delta_E(y)^{qv-(m-d)} \, d\mu(y),
\end{aligned} \tag{6.158}$$

where the last step in (6.158) uses the inequality $\sigma(Q \cap \pi_y^{\widetilde{\kappa}}) \leq C\delta_E(y)^d$ which, in turn, is a consequence of Lemma 6.3, the fact that $(E, \rho\big|_E, \sigma)$ is a d-dimensional ADR space, and the observation that $\delta_E(y) \leq \mathrm{dist}_{\rho_\#}(y, Q) \leq C\,\ell(Q) \leq C\,\mathrm{diam}_\rho(E)$ on the domain of integration (of the third integral in (6.158)). Moreover, reasoning as in (3.98) we obtain

$$\int_{B_{\rho_\#}(x_Q, C\ell(Q))} |(\Theta_i 1)(y)|^q \delta_E(y)^{qv-(m-d)} \, d\mu(y) \leq C i^{2qv} \ell(Q)^d \leq C i^{2qv} \sigma(Q) \tag{6.159}$$

for every $Q \in \mathbb{D}(E)$, so by combining (6.158) and (6.159) we arrive at the conclusion that

$$\tfrac{1}{\sigma(Q)} \int_Q (S^i_{Q,\widetilde{\kappa}}(x))^q \, d\sigma(x) \leq C(i) < \infty, \tag{6.160}$$

for each cube $Q \in \mathbb{D}(E)$ and each $i \in \mathbb{N}$. Thus, in particular, $A^i < \infty$ for every $i \in \mathbb{N}$.

At this stage in the proof, the incisive step is the claim that, in fact,

$$\begin{gathered}
\exists\, A \in (0,\infty) \quad \text{independent of } i \text{ such that} \\
\tfrac{1}{\sigma(Q)} \textstyle\int_Q (S^i_{Q,\widetilde{\kappa}}(x))^q \, d\sigma(x) \leq A, \quad \forall\, Q \in \mathbb{D}(E).
\end{gathered} \tag{6.161}$$

In the process of proving this claim we shall show that one can take $A := \beta^{-1}(C + N^q)$ where $C \in (0,\infty)$ is a constant which depends only on geometry, the estimates satisfied by θ, and κ. To get started, fix $i \in \mathbb{N}$ and first observe that if $Q \in \mathbb{D}(E)$ is such that $\Omega_Q^{N,i} = \emptyset$, then $S^i_{Q,\widetilde{\kappa}} \leq S^i_Q \leq N$ on Q, hence for such Q's (6.161) will hold if we impose the condition that $A \geq N^q$. Next, let $Q \in \mathbb{D}(E)$ be such that $\Omega_Q^{N,i} \neq \emptyset$. Then, thanks to (6.156), it follows that $\Omega_Q^{N,i}$ is an open, nonempty, proper subset of Q. Recall from (2.57) that $\big(Q, \rho|_Q, \sigma\lfloor Q\big)$ is a space of homogeneous type and the doubling constant of the measure $\sigma\lfloor Q$ is independent of Q. Then there exists a Whitney decomposition of $\Omega_Q^{N,i}$ relative to Q via dyadic cubes $\{Q_k\}_{k\in I_Q^{N,i}}$

as described in Lemma 6.14 (used with $\mathcal{O} := Q$ and $\Omega := \Omega_Q^{N,i}$). Introducing $F_Q^{N,i} := Q \setminus \Omega_Q^{N,i}$ we may then write

$$\int_Q (S^i_{Q,\widetilde{\kappa}}(x))^q \, d\sigma(x) = \int_{F_Q^{N,i}} (S^i_{Q,\widetilde{\kappa}}(x))^q \, d\sigma(x) + \sum_{k\in I_Q^{N,i}} \int_{Q_k} (S^i_{Q,\widetilde{\kappa}}(x))^q \, d\sigma(x) =: I + II. \tag{6.162}$$

Since $\widetilde{\kappa} < \kappa$ forces $S^i_{Q,\widetilde{\kappa}} \leq S^i_Q \leq N$ on $F_Q^{N,i}$, we further have

$$I \leq \int_{F_Q^{N,i}} (S^i_Q(x))^q \, d\sigma(x) \leq N^q \sigma(Q). \tag{6.163}$$

To estimate II, we write

$$\begin{aligned} II = {} & \sum_{k\in I_Q^{N,i}} \int_{Q_k} (S^i_{Q_k,\widetilde{\kappa}}(x))^q \, d\sigma(x) \\ & + \sum_{k\in I_Q^{N,i}} \int_{Q_k} \Big(\int\limits_{\substack{y\in\Gamma_{\widetilde{\kappa}}(x) \\ \eta\ell(Q_k)\leq\rho_\#(y,x)<\eta\ell(Q)}} |(\Theta_i 1)(y)|^q \delta_E(y)^{q\upsilon-m} d\mu(y) \Big) \, d\sigma(x) \\ =: {} & III + IV. \end{aligned} \tag{6.164}$$

By recalling (6.157), the fact that the family $\{Q_k\}_{k\in I_Q^{N,i}}$ consists of pairwise disjoint cubes from $\mathbb{D}(E)$ contained in $\Omega_Q^{N,i}$, as well as assumption (6.153), we have

$$III \leq \sum_{k\in I_Q^{N,i}} A^i \sigma(Q_k) \leq A^i \sigma\big(\Omega_Q^{N,i}\big) \leq A^i(1-\beta)\sigma(Q). \tag{6.165}$$

Moving on, from (3.2) and (3.19) (given that $\upsilon - a > 0$) we see that $|(\Theta_i 1)(y)| \leq \frac{C}{\delta_E(y)^\upsilon}$ for every $y \in \mathscr{X} \setminus E$. Thus, if $C_0 > 0$ is some large finite fixed constant which will be specified later (just below (6.171), to be precise), and if $k \in I_Q^{N,i}$, then for each $x \in Q_k$ there holds

$$\begin{aligned} & \int\limits_{\substack{y\in\Gamma_{\widetilde{\kappa}}(x) \\ \eta\ell(Q_k)\leq\rho_\#(x,y)\leq C_0\ell(Q_k)}} |(\Theta_i 1)(y)|^q \delta_E(y)^{q\upsilon-m} \, d\mu(y) \leq C \int\limits_{\substack{y\in\Gamma_{\widetilde{\kappa}}(x) \\ \eta\ell(Q_k)\leq\rho_\#(x,y)\leq C_0\ell(Q_k)}} \frac{d\mu(y)}{\delta_E(y)^m} \\ & \qquad \leq C\ell(Q_k)^{-m} \mu\Big(\big\{ y \in \Gamma_{\widetilde{\kappa}}(x) : \eta\ell(Q_k) \leq \rho_\#(x,y) \leq C_0\ell(Q_k) \big\} \Big) \leq C < \infty, \end{aligned} \tag{6.166}$$

for some $C > 0$ independent of x, k, Q and i. In turn, (6.166) entails

$$\begin{aligned} & \sum_{k\in I_Q^{N,i}} \int_{Q_k} \Big(\int\limits_{\substack{y\in\Gamma_{\widetilde{\kappa}}(x) \\ \eta\ell(Q_k)\leq\rho_\#(x,y)\leq C_0\ell(Q_k)}} |(\Theta_i 1)(y)|^q \delta_E(y)^{q\upsilon-m} \, d\mu(y) \Big) \, d\sigma(x) \\ & \qquad\qquad \leq C \sum_{k\in I_Q^{N,i}} \sigma(Q_k) \leq C\sigma\big(\Omega_Q^{N,i}\big) \leq C\sigma(Q), \end{aligned} \tag{6.167}$$

which once again suits our purposes. Next, since $\{Q_k\}_{k\in I_Q^{N,i}}$ is a Whitney decomposition of $\Omega_Q^{N,i}$ relative to Q, for each $k \in I_Q^{N,i}$ there exists $x_k \in F_Q^{N,i}$ such that

$$\mathrm{dist}_{\rho_\#}(x_k, Q_k) \leq c\,\ell(Q_k), \tag{6.168}$$

for some finite $c > 0$ independent of k, Q and i. We now claim that there exits $\widetilde{\kappa} \in (0, \kappa)$ depending on the constants associated with the Whitney decomposition of $\Omega_Q^{N,i}$ (hence, ultimately, on finite positive geometric constants associated with $(E, \rho\big|_E, \sigma)$), as well as on κ and the constant C_0, but independent of k, Q and i, such that

$$x \in Q_k,\ \ y \in \Gamma_{\widetilde{\kappa}}(x) \ \text{ and } \ C_0\ell(Q_k) < \rho_\#(x,y) \Longrightarrow y \in \Gamma_\kappa(x_k). \tag{6.169}$$

To justify this claim, suppose that $\widetilde{\kappa} \in (0, \kappa)$ and fix $x \in Q_k$ along with $y \in \Gamma_{\widetilde{\kappa}}(x)$ such that $C_0\ell(Q_k) < \mathrm{dist}_{\rho_\#}(y, Q)$. Then

$$C_0\ell(Q_k) < \rho_\#(y,x) < (1+\widetilde{\kappa})\delta_E(y) < (1+\kappa)\delta_E(y). \tag{6.170}$$

Also, if we choose a finite number $\vartheta \in \big(0, (\log_2 C_\rho)^{-1}\big]$, Theorem 2.2 gives that $(\rho_\#)^\vartheta$ is a genuine distance. As such, we may estimate based on (6.168), (6.170) and hypotheses

$$\begin{aligned}
\rho_\#(y,x_k)^\vartheta &\leq \rho_\#(y,x)^\vartheta + \rho_\#(x,x_k)^\vartheta < (1+\widetilde{\kappa})^\vartheta\delta_E(y)^\vartheta + c^\vartheta\ell(Q_k)^\vartheta \\
&\leq (1+\widetilde{\kappa})^\vartheta\delta_E(y)^\vartheta + c^\vartheta\frac{(1+\kappa)^\vartheta}{C_0^\vartheta}\delta_E(y)^\vartheta \\
&\leq (1+\kappa)^\vartheta\delta_E(y)^\vartheta,
\end{aligned} \tag{6.171}$$

provided $C_0 > c\Big[1-\Big(\frac{1}{1+\kappa}\Big)^\vartheta\Big]^{-1/\vartheta}$ and $0 < \widetilde{\kappa} < (1+\kappa)\Big[1-\Big(\frac{c}{C_0}\Big)^\vartheta\Big]^{1/\vartheta} - 1$. Assuming that this is the case, (6.169) now follows from (6.171).

Going further, with (6.169) in hand and upon recalling that $S_Q^i(x_k) \leq N$, we may estimate

$$\begin{aligned}
&\sum_{k\in I_Q^{N,i}}\int_{Q_k}\Big(\int\limits_{\substack{y\in\Gamma_{\widetilde{\kappa}}(x)\\ C_0\ell(Q_k)<\rho_\#(x,y)<\eta\ell(Q)}} |(\Theta_i 1)(y)|^q\delta_E(y)^{qv-m}\,d\mu(y)\Big)\,d\sigma(x) \\
&\qquad \leq \sum_{k\in I_Q^{N,i}}\int_{Q_k}\Big(\int\limits_{\substack{y\in\Gamma_\kappa(x_k)\\ \rho_\#(x,y)<\eta\ell(Q)}} |(\Theta_i 1)(y)|^q\delta_E(y)^{qv-m}\,d\mu(y)\Big)\,d\sigma(x) \\
&\qquad = \sum_{k\in I_Q^{N,i}}\int_{x\in Q_k}(S_Q^i(x_k))^q\,d\sigma(x) \leq N^q\sum_{k\in I_Q^{N,i}}\sigma(Q_k) \leq N^q\sigma(Q),
\end{aligned} \tag{6.172}$$

which is of the right order. In concert, (6.167)-(6.172) prove that there exists $C \in (0,\infty)$ depending only on geometry, the estimates satisfied by the kernel θ, and κ, with the property that $IV \leq (C+N^q)\sigma(Q)$. In combination with (6.162)-(6.165), this then allows us to conclude that

$$\int_Q (S_{Q,\widetilde{\kappa}}^i(x))^q d\sigma \leq A^i(1-\beta)\sigma(Q) + (C+N^q)\sigma(Q), \qquad \forall\, Q \in \mathbb{D}(E). \tag{6.173}$$

In particular, if we divide (6.173) by $\sigma(Q)$, then take the supremum over $Q \in \mathbb{D}(E)$ we arrive at the conclusion that $A^i \leq A^i(1-\beta) + C$ for each $i \in \mathbb{N}$. Upon recalling that $A^i \in (0,\infty)$ for each $i \in \mathbb{N}$ and that $\beta \in (0,1)$, it follows from this that $\sup_{i\in\mathbb{N}} A^i \leq \beta^{-1}(C + N^q) < \infty$. Hence, (6.161) is true.

Consider now the function $S_{Q,\widetilde{\kappa}}$ defined analogously to S_Q but with $\widetilde{\kappa}$ in place of κ. Given that $\lim\limits_{i\to\infty} S^i_{Q,\widetilde{\kappa}} = S_{Q,\widetilde{\kappa}}$ pointwise in E, from (6.161) and Lebesgue's Monotone Convergence Theorem we may conclude that

$$\exists\, C \in (0,\infty) \quad \text{such that} \quad \tfrac{1}{\sigma(Q)} \int_Q (S_{Q,\widetilde{\kappa}}(x))^q \, d\sigma(x) \leq C, \qquad \forall\, Q \in \mathbb{D}(E). \tag{6.174}$$

Next, observe that

$$x, y \in B_{\rho_\#}\big(x_Q, \eta C_\rho^{-1}\ell(Q)\big) \implies \rho_\#(x,y) \leq \eta\ell(Q). \tag{6.175}$$

Then, based on (6.152), (6.175), (6.58), (ii) in Lemma 6.2, (6.19), and the fact that $\big(E, \rho\big|_E, \sigma\big)$ is a d-dimensional ADR space, we may estimate (using notation introduced in (6.98)):

$$\begin{aligned}
\int\limits_{\Delta(x_Q,\eta C_\rho^{-1}\ell(Q))} & (S_{Q,\widetilde{\kappa}}(x))^q \, d\sigma(x) \\
&= \int\limits_{\Delta(x_Q,\eta C_\rho^{-1}\ell(Q))} \Big(\int\limits_{\substack{y\in\Gamma_{\widetilde{\kappa}}(x) \\ \rho_\#(x,y)<\eta\ell(Q)}} |(\Theta 1)(y)|^q \delta_E(y)^{q\upsilon - m} \, d\mu(y)\Big) d\sigma(x) \\
&\geq \int\limits_{\Delta(x_Q,\eta C_\rho^{-1}\ell(Q))} \Big(\int\limits_{\substack{y\in\Gamma_{\widetilde{\kappa}}(x) \\ \rho_\#(y,x_Q)<\eta C_\rho^{-1}\ell(Q)}} |(\Theta 1)(y)|^q \delta_E(y)^{q\upsilon - m} \, d\mu(y)\Big) d\sigma(x) \\
&= \int\limits_{\substack{y\in\mathcal{F}_{\widetilde{\kappa}}(\Delta(x_Q,\eta C_\rho^{-1}\ell(Q))) \\ \rho_\#(y,x_Q)<\eta C_\rho^{-1}\ell(Q)}} |(\Theta 1)(y)|^q \delta_E(y)^{q\upsilon - m} \sigma\big(\Delta(x_Q,\eta C_\rho^{-1}\ell(Q)) \cap \pi_y^{\widetilde{\kappa}}\big) \, d\mu(y) \\
&\geq \int\limits_{\substack{y\in\mathcal{T}_{\widetilde{\kappa}}(\Delta(x_Q,\eta C_\rho^{-1}\ell(Q))) \\ \rho_\#(y,x_Q)<\eta C_\rho^{-1}\ell(Q)}} |(\Theta 1)(y)|^q \delta_E(y)^{q\upsilon - m} \sigma\big(\Delta(x_Q,\eta C_\rho^{-1}\ell(Q)) \cap \pi_y^{\widetilde{\kappa}}\big) \, d\mu(y) \\
&= \int\limits_{\substack{y\in\mathcal{T}_{\widetilde{\kappa}}(\Delta(x_Q,\eta C_\rho^{-1}\ell(Q))) \\ \rho_\#(y,x_Q)<\eta C_\rho^{-1}\ell(Q)}} |(\Theta 1)(y)|^q \delta_E(y)^{q\upsilon - m} \sigma\big(\pi_y^{\widetilde{\kappa}}\big) \, d\mu(y) \\
&\approx \int\limits_{\substack{y\in\mathcal{T}_{\widetilde{\kappa}}(\Delta(x_Q,\eta C_\rho^{-1}\ell(Q))) \\ \rho_\#(y,x_Q)<\eta C_\rho^{-1}\ell(Q)}} |(\Theta 1)(y)|^q \delta_E(y)^{q\upsilon - (m-d)} \, d\mu(y),
\end{aligned} \tag{6.176}$$

uniformly for $Q \in \mathbb{D}(E)$. Let us also observe that there exists an integer $M_o \in \mathbb{N}$ (depending only on geometry) with the property that for every $Q \in \mathbb{D}(E)$ the ball $\Delta(x_Q, \eta C_\rho^{-1}\ell(Q))$ may

be covered by at most M_o dyadic cubes of the same generation as Q, and that for every such cube $\widetilde{Q}$ there holds $S_{\widetilde{Q}} = S_Q$. Having noticed this, we then deduce from (6.174), (6.176), and (6.23) that there exists $C \in (0,\infty)$ satisfying

$$\frac{1}{\sigma(Q)} \int\limits_{B_{\rho_\#}(x_Q,\,\eta C_\rho^{-2}\ell(Q))\setminus E} |(\Theta 1)(x)|^q \delta_E(x)^{q\upsilon-(m-d)}\, d\mu(x) \leq C, \qquad \forall\, Q \in \mathbb{D}(E). \tag{6.177}$$

With this in hand, (6.154) now follows with the help of (2.145), if η is sufficiently large to begin with (depending only on geometry). $\square$

Our last auxiliary result is an estimate of geometrical nature, on a nontangential approach region. For a proof (and for more general results of this type) see [61].

Lemma 6.16. *Let $(\mathscr{X},\rho,\mu)$ be an m-dimensional* ADR *space for some $m>0$. Assume that E is a closed subset of $(\mathscr{X},\tau_\rho)$ with the property that there exists a Borel measure σ on $(E,\tau_{\rho|_E})$ such that $\big(E,\rho\big|_E,\sigma\big)$ is a d-dimensional* ADR *space for some $d\geq 0$. Then for each $\kappa>0$, $\beta<m$, $M>m-\beta$, there exists a finite constant $C>0$ depending on κ, M, β, and the* ADR *constants of $\mathscr{X}$ and E, such that*

$$\int_{\Gamma_\kappa(z)} \frac{\delta_E(x)^{-\beta}}{\rho_\#(x,y)^M}\, d\mu(x) \leq C\rho(y,z)^{m-\beta-M}, \quad \textit{for all }\ z,y\in E \textit{ with } z\neq y. \tag{6.178}$$

Now we are ready to proceed with the

Proof of Proposition 6.13. Based on Lemma 6.15, it suffices to prove that if the hypotheses of Proposition 6.13 are satisfied, then there exist $N<\infty$ and $\beta\in(0,1)$ such that (6.153) holds. To this end, let $N>0$ be a large finite constant, to be specified later, and fix an arbitrary $Q\in\mathbb{D}(E)$. Also, recall c_q from (6.103) and fix an arbitrary number $\eta>0$. Then, with S_Q as in (6.152) and some finite constant $c>0$ to be specified later, we may write

$$\begin{aligned}
&\sigma\Big(\big\{x\in Q:\ S_Q(x)>N\big\}\Big)\\
&\qquad \leq \sigma\Big(\big\{x\in Q:\ \Big(\int\limits_{y\in\Gamma_\kappa(x),\,\rho_\#(x,y)<\eta\ell(Q)} |(\Theta\mathbf{1}_{cQ})(y)|^q\delta_E(y)^{q\upsilon-m}\, d\mu(y)\Big)^{\frac{1}{q}}>N/2\big\}\Big)\\
&\qquad\quad +\sigma\Big(\big\{x\in Q:\ \Big(\int\limits_{y\in\Gamma_\kappa(x),\,\rho_\#(x,y)<\eta\ell(Q)} |(\Theta\mathbf{1}_{E\setminus cQ})(y)|^q\delta_E(y)^{q\upsilon-m}\, d\mu(y)\Big)^{\frac{1}{q}}>N/2\big\}\Big)\\
&\qquad =: I+II,
\end{aligned} \tag{6.179}$$

where we have used the notation $c\,Q := E\cap B_{\rho_\#}\big(x_Q, c\,\ell(Q)\big)$. Note that under the assumption (6.149) (and the fact that σ is doubling) we may estimate

$$I\leq \sigma\Big(\big\{x\in Q:\ \Big(\int_{\Gamma_\kappa(x)} |(\Theta\mathbf{1}_{cQ})(y)|^q\delta_E(y)^{q\upsilon-m}\, d\mu(y)\Big)^{\frac{1}{q}}>N/2\big\}\Big)\leq \tfrac{C}{N^p}\sigma(Q), \tag{6.180}$$

which suits our purposes.

Going further, select some finite constant $c_o \geq \sup_{Q' \in \mathbb{D}(E)} \big(\mathrm{diam}_{\rho_\#}(Q')/\ell(Q')\big)$. Given $x \in Q$ fixed, note that for each $y \in B_{\rho_\#}(x, \eta\ell(Q))$ we have

$$\begin{aligned} \rho_\#(y, x_Q) &\leq C_{\rho_\#} \max\{\rho_\#(y,x), \rho_\#(x, x_Q)\} && (6.181) \\ &\leq C_\rho \max\{\eta, c_o\}\, \ell(Q) \leq c^{-1} C_\rho \max\{\eta, c_o\}\, \rho_\#(z, x_Q), \qquad \forall\, z \in E \setminus \mathbf{1}_{c\,Q}. \end{aligned}$$

Consequently, if $z \in E \setminus \mathbf{1}_{c\,Q}$, then

$$\begin{aligned} \rho_\#(z, x_Q) &\leq C_{\rho_\#} \max\{\rho_\#(z,y), \rho_\#(y, x_Q)\} && (6.182) \\ &\leq C_\rho \rho_\#(z,y) + c^{-1} C_\rho^2 \max\{\eta, c_o\}\, \rho_\#(z, x_Q), \quad \forall\, y \in B_{\rho_\#}(x, \eta\ell(Q)). \end{aligned}$$

Hence, choosing the finite constant $c > 0$ sufficiently large so that $c^{-1} C_\rho^2 \max\{\eta, c_o\} < \frac{1}{2}$ forces $\rho_\#(z, x_Q) \leq 2C_\rho \rho_\#(z,y)$ for all $y \in B_{\rho_\#}(x, \eta\ell(Q))$. Making use of this, (3.2), and (3.20) we may then write

$$\begin{aligned} |(\Theta \mathbf{1}_{E\setminus c\,Q})(y)| &\leq C \int_{E\setminus c\,Q} \frac{\delta_E(y)^{-a}}{\rho_\#(z,y)^{d+\upsilon-a}}\, d\sigma(z) \leq C\delta_E(y)^{-a} \int_{z\in E,\, \rho_\#(z,x_Q) > c\,\ell(Q)} \frac{d\sigma(z)}{\rho_\#(z,x_Q)^{d+\upsilon-a}} \\ &\leq C \frac{\delta_E(y)^{-a}}{\ell(Q)^{\upsilon-a}}, \qquad \forall\, y \in B_{\rho_\#}(x, \eta\ell(Q)). && (6.183) \end{aligned}$$

Pick now $1 < c_1 < c_2 < c$ such that there exists $w \in c_2 Q \setminus c_1 Q$ (which may be assured by further increasing c if needed, given that $(E, \rho|_E, \sigma)$ is a d-dimensional ADR space). Then clearly $\rho_\#(x,w) \approx \ell(Q)$ and we claim that also

$$\rho_\#(y,w) \approx \ell(Q), \qquad \text{uniformly for } \; y \in \Gamma_\kappa(x) \cap B_{\rho_\#}(x, \eta\ell(Q)). \tag{6.184}$$

Indeed, on the one hand, if the point $y \in \mathscr{X}$ is such that $\rho_\#(y,x) < \eta\ell(Q)$ then we obtain $\rho_\#(y,w) \leq C_\rho \max\{\rho_\#(y,x), \rho_\#(x,w)\} \leq C\ell(Q)$. On the other hand, if we additionally know that $y \in \Gamma_\kappa(x)$, then $\rho_\#(y,x) < (1+\kappa)\delta_E(y) \leq (1+\kappa)\rho_\#(y,w)$, hence

$$\begin{aligned} C\ell(Q) &\leq \rho_\#(x,w) \leq C_{\rho_\#} \max\{\rho_\#(x,y), \rho_\#(y,w)\} \\ &\leq C_\rho (1+\kappa) \rho_\#(y,w) \leq C\ell(Q), && (6.185) \end{aligned}$$

proving (6.184).

Select now a real number $M > q(\upsilon - a)$. Combining (6.183) and (6.184) we then obtain

$$\begin{aligned} &\int_{\substack{y\in\Gamma_\kappa(x) \\ \rho_\#(x,y)<\eta\ell(Q)}} |(\Theta \mathbf{1}_{E\setminus c\,Q})(y)|^q \delta_E(y)^{q\upsilon - m}\, d\mu(y) && (6.186) \\ &\leq C \int_{\substack{y\in\Gamma_\kappa(x) \\ \rho_\#(x,y)<\eta\ell(Q)}} \frac{1}{\ell(Q)^{q(\upsilon-a)}} \cdot \delta_E(y)^{q(\upsilon-a)-m}\, d\mu(y) \\ &\leq C\ell(Q)^{M-q(\upsilon-a)} \int_{\Gamma_\kappa(x)} \frac{\delta_E(y)^{-[m-q(\upsilon-a)]}}{\rho_\#(y,w)^M}\, d\mu(y) \\ &\leq C\ell(Q)^{M-q(\upsilon-a)} \rho_\#(x,w)^{-M+q(\upsilon-a)} \leq C, \qquad \forall\, x \in Q, \end{aligned}$$

where for the penultimate inequality in (6.186) we have relied on Lemma 6.16 (used here with $\beta := m - q(\upsilon - a)$).

With this in hand, we are now ready to estimate the term II (appearing in (6.179)). Concretely, applying first Tschebyshev's inequality and then invoking (6.186) we obtain

$$II \leq \tfrac{C}{N} \int_Q \Big(\int\limits_{\substack{y \in \Gamma_\kappa(x) \\ \rho_\#(x,y) < \eta \ell(Q)}} |(\Theta \mathbf{1}_{E \setminus cQ})(y)|^q \frac{d\mu(y)}{\delta_E(y)^{m-q\upsilon}} \Big)^{\frac{1}{q}} d\sigma(x) \leq \tfrac{C}{N} \sigma(Q). \tag{6.187}$$

Combining (6.179), (6.180) and (6.187) we see that, for each $\beta \in (0,1)$, if we choose $N > 0$ sufficiently large, then

$$\sigma\Big(\big\{x \in Q : S_Q(x) > N\big\}\Big) \leq \tfrac{C}{N^{\min\{1,p\}}} \sigma(Q) < (1-\beta)\sigma(Q), \qquad \forall\, Q \in \mathbb{D}(E). \tag{6.188}$$

Hence, (6.153) holds and the proof of the proposition is complete. □

6.4 Extrapolating square function estimates

We now combine our results to prove two extrapolation theorems for square function estimates associated with integral operators Θ_E, as defined in Section 3. First, we use Theorem 6.10 to prove the extrapolation result in Theorem 6.18, and then we combine this with Theorem 6.12 to obtain another extrapolation result in Theorem 6.20.

In the first part of this subsection we digress to clarify terminology and background results concerning the scale of Hardy spaces H^p for $p \in (0, \infty)$ in the context of a d-dimensional Ahlfors-David Regular space. In particular, we consider an atomic characterization for these spaces based on the work of R.R. Coifman and G. Weiss in [19], as well as a maximal function characterization based on the work of R.A. Macías and C. Segovia in [57]. The theory of Hardy spaces in the context considered here has also been developed by D. Mitrea, I. Mitrea, M. Mitrea and S. Monniaux in [60].

Consider a d-dimensional ADR space (E, ρ, σ) and let $\beta \in (0, \infty)$. Given a real-valued function f on E, define its `Hölder semi-norm` (of order β, relative to the quasi-distance ρ) by setting

$$\|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)} := \sup_{x,y \in E,\, x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\beta}, \tag{6.189}$$

and define the homogeneous Hölder space $\dot{\mathscr{C}}^\beta(E)$ as

$$\dot{\mathscr{C}}^\beta(E,\rho) := \big\{ f : E \to \mathbb{R} : \|f\|_{\dot{\mathscr{C}}^\beta(E,\rho)} < \infty \big\}. \tag{6.190}$$

Going further, set $\dot{\mathscr{C}}^\beta_c(E,\rho)$ for the subspace of $\dot{\mathscr{C}}^\beta(E,\rho)$ consisting of functions which vanish identically outside a bounded set. Then define the class of `test functions` on E as

$$\mathscr{D}(E,\rho) := \bigcap_{0 < \beta < [\log_2 C_\rho]^{-1}} \dot{\mathscr{C}}^\beta_c(E,\rho), \tag{6.191}$$

equipped with a certain topology, $\tau_{\mathscr{D}}$, which we shall describe next. Specifically, fix a nested family $\{K_n\}_{n \in \mathbb{N}}$ of ρ-bounded subsets of E with the property that any ρ-ball is contained in

one of the K_n's. Then, for each $n \in \mathbb{N}$, denote by $\mathscr{D}_n(E,\rho)$ the collection of functions from $\mathscr{D}(E,\rho)$ which vanish in $E \setminus K_n$. With $\|\cdot\|_\infty$ standing for the supremum norm on E, this becomes a Frechét space when equipped with the topology τ_n induced by the family of norms

$$\big\{\|\cdot\|_\infty + \|\cdot\|_{\dot{\mathscr{C}}^\beta(E,\rho)} : \beta \text{ rational number such that } 0 < \beta < [\log_2 C_\rho]^{-1}\big\}. \tag{6.192}$$

That is, $\mathscr{D}_n(E,\rho)$ is a Hausdorff topological space, whose topology is induced by a countable family of semi-norms, and which is complete (as a uniform space with the uniformity canonically induced by the aforementioned family of semi-norms or, equivalently, as a metric space when endowed with a metric yielding the same topology as τ_n). Since for any $n \in \mathbb{N}$ the topology induced by τ_{n+1} on $\mathscr{D}_n(X,\rho)$ coincides with τ_n, we may turn $\mathscr{D}(X,\rho)$ into a topological space, $(\mathscr{D}(X,\rho), \tau_{\mathscr{D}})$, by regarding it as the strict inductive limit of the family of topological spaces $\big\{(\mathscr{D}_n(X,\rho),\tau_n)\big\}_{n\in\mathbb{N}}$. Having accomplished this, we then define the `space of distributions` $\mathscr{D}'(E,\rho)$ on E as the (topological) dual of $\mathscr{D}(E,\rho)$, and denote by $\langle\cdot,\cdot\rangle$ the natural duality pairing between distributions in $\mathscr{D}'(E,\rho)$ and test functions in $\mathscr{D}(E,\rho)$.

To proceed, for each number $\gamma \in \big(0, \big[\log_2 C_\rho\big]^{-1}\big)$ and each point $x \in E$ define the class $\mathcal{B}^\gamma_\rho(x)$ of (ρ,γ)-`normalized bump-functions supported near` x by

$$\begin{aligned}\mathcal{B}^\gamma_\rho(x) := \Big\{\psi \in \mathscr{D}(E,\rho) : \exists\, r > 0 \text{ such that } \psi = 0 \text{ on } E \setminus B_\rho(x,r) \text{ and}\\ \|\psi\|_\infty + r^\gamma \|\psi\|_{\dot{\mathscr{C}}^\gamma(E,\rho)} \leq r^{-d}\Big\}.\end{aligned} \tag{6.193}$$

In this setting, define the `grand maximal function` of a distribution $f \in \mathscr{D}'(E,\rho)$ by setting (with the duality paring understood as before)

$$f^*_{\rho,\gamma}(x) := \sup_{\psi \in \mathcal{B}^\gamma_\rho(x)} \big|\langle f,\psi\rangle\big|, \qquad \forall\, x \in E. \tag{6.194}$$

Given an exponent p satisfying

$$\frac{d}{d + [\log_2 C_\rho]^{-1}} < p < \infty, \tag{6.195}$$

define the `Hardy space` $H^p(E,\rho,\sigma)$ by setting

$$\begin{aligned}H^p(E,\rho,\sigma) := \Big\{f \in \mathscr{D}'(E,\rho) : \forall\, \gamma \in \mathbb{R} \text{ so that } d\big(\tfrac{1}{p} - 1\big) < \gamma < [\log_2 C_\rho]^{-1}\\ \text{it follows that } f^*_{\rho_\#,\gamma} \in L^p(E,\sigma)\Big\}.\end{aligned} \tag{6.196}$$

A closely related version of the above Hardy space is $\widetilde{H}^p(E,\rho,\sigma)$, with p as before, defined as

$$\begin{aligned}\widetilde{H}^p(E,\rho,\sigma) := \Big\{f \in \mathscr{D}'(E,\rho) : \exists\, \gamma \in \mathbb{R} \text{ so that } d\big(\tfrac{1}{p} - 1\big) < \gamma < [\log_2 C_\rho]^{-1}\\ \text{and with the property that } f^*_{\rho_\#,\gamma} \in L^p(E,\sigma)\Big\}.\end{aligned} \tag{6.197}$$

Moving on, given an index

$$\frac{d}{d + [\log_2 C_\rho]^{-1}} < p \leq 1, \tag{6.198}$$

call a function $a \in L^\infty(E,\sigma)$ a p-**atom** provided there exist $x_0 \in E$ and a real number $r > 0$ with the property that

$$\operatorname{supp} a \subseteq E \cap B_\rho(x_0, r), \quad \|a\|_{L^\infty(E,\sigma)} \leq r^{-d/p}, \quad \int_E a\, d\sigma = 0. \tag{6.199}$$

In the case when E is bounded we also agree to consider the constant function $\sigma(E)^{-1/p}$ as a p-atom. Then, for each p as in (6.198), define the **atomic Hardy space** $H^p_{at}(E,\rho,\sigma)$ as

$$H^p_{at}(E,\rho,\sigma) := \Big\{ f \in \big(\dot{\mathscr{C}}^{d(1/p-1)}(E,\rho)\big)^* : \exists\, \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N}) \text{ and } p\text{-atoms } \{a_j\}_{j\in\mathbb{N}} \\ \text{such that } f = \sum_{j\in\mathbb{N}} \lambda_j a_j \ \text{ in } \ \big(\dot{\mathscr{C}}^{d(1/p-1)}(E,\rho)\big)^* \Big\}, \tag{6.200}$$

and equip this space with the quasi-norm $\|\cdot\|_{H^p_{at}(E,\rho,\sigma)}$ defined for each $f \in H^p_{at}(E,\rho,\sigma)$ by

$$\|f\|_{H^p_{at}(E,\rho,\sigma)} := \inf \Big\{ \Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p \Big)^{1/p} : f = \sum_{j\in\mathbb{N}} \lambda_j a_j \ \text{ as in (6.200)} \Big\}. \tag{6.201}$$

The following atomic decomposition theorem, extending work in [57], has been established in [60].

Theorem 6.17. *Assume that (E,ρ,σ) is a d-dimensional* ADR. *Then*

$$H^p(E,\rho,\sigma) = \widetilde{H}^p(E,\rho,\sigma) = L^p(E,\sigma) \quad \textit{for each } p \in (1,\infty). \tag{6.202}$$

Suppose now that p is as in (6.198) *and, for every functional $f \in H^p_{at}(E,\rho,\sigma)$, denote by $\widetilde{f}$ the distribution in $\mathscr{D}'(E,\rho)$ defined as the restriction of f to $\mathscr{D}(E,\rho)$. Then the assignment $f \mapsto \widetilde{f}$ induces a well-defined, injective linear mapping from $H^p_{at}(E,\rho,\sigma)$ onto the space $\widetilde{H}^p(E,\rho,\sigma)$. Moreover, for each*

$$\gamma \in \mathbb{R} \quad \textit{with} \quad d\big(\tfrac{1}{p} - 1\big) < \gamma < [\log_2 C_\rho]^{-1} \tag{6.203}$$

there exist two finite constants $c_1, c_2 > 0$ such that

$$c_1 \|f\|_{H^p_{at}(E,\rho,\sigma)} \leq \|(\widetilde{f})^*_{\rho_\#,\gamma}\|_{L^p(E,\sigma)} \leq c_2 \|f\|_{H^p_{at}(E,\rho,\sigma)} \quad \textit{for all } f \in H^p_{at}(E,\rho,\sigma). \tag{6.204}$$

Consequently, the spaces $H^p(E,\rho,\sigma)$, $\widetilde{H}^p(E,\rho,\sigma)$ are naturally identified with $H^p_{at}(E,\rho,\sigma)$. In particular, they do not depend on the particular choice of the index γ as in (6.203).

As a corollary, whenever (6.203) *holds one can find a finite constant $c = c(p,\rho,\gamma) > 0$ such that for every distribution $f \in \mathscr{D}'(E,\rho)$ with the property that its grand maximal function $f^*_{\rho_\#,\gamma}$ belongs to $L^p(E,\sigma)$ there exist a sequence of p-atoms $\{a_j\}_{j\in\mathbb{N}}$ on X and a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ for which*

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \textit{in } \ \mathscr{D}'(E,\rho) \tag{6.205}$$

and

$$\Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p \Big)^{1/p} \leq c \|f^*_{\rho_\#,\gamma}\|_{L^p(E,\sigma)}. \tag{6.206}$$

Finally, whenever (6.203) *holds one can find a finite constant* $c' = c'(p,\rho,\gamma) > 0$ *such that, given a distribution* $f \in \mathscr{D}'(E,\rho)$*, a sequence of* p*-atoms* $\{a_j\}_{j\in\mathbb{N}}$*, and a numerical sequence* $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ *such that* (6.205) *holds, then*

$$\|f^*_{\rho_\#,\gamma}\|_{L^p(E,\sigma)} \leq c'\Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Big)^{1/p}. \tag{6.207}$$

Consider now the setting of Section 3.1 and suppose that θ is a function as in (3.1) which satisfies (3.2) and such that there exists $\alpha \in (0,\infty)$ with the property that for all $x \in \mathscr{X} \setminus E$ and $y \in E$ there holds

$$\begin{aligned} &|\theta(x,y) - \theta(x,\widetilde{y})| \leq C_\theta \frac{\rho(y,\widetilde{y})^\alpha}{\rho(x,y)^{d+\upsilon+\alpha}} \Big(\frac{\mathrm{dist}_\rho(x,E)}{\rho(x,y)}\Big)^{-a}, \\ &\qquad \forall\, \widetilde{y} \in E \ \text{ with } \ \rho(y,\widetilde{y}) \leq \tfrac{1}{2}\rho(x,y). \end{aligned} \tag{6.208}$$

We are now ready to present the first main result in this subsection.

Theorem 6.18. *Let* d, m *be two real numbers such that* $0 < d < m$*. Assume that* $(\mathscr{X},\rho,\mu)$ *is an* m*-dimensional* ADR *space,* E *is a closed subset of* $(\mathscr{X},\tau_\rho)$*, and* σ *is a Borel measure on* $(E,\tau_{\rho|_E})$ *with the property that* $(E,\rho\big|_E,\sigma)$ *is a* d*-dimensional* ADR *space.*

Furthermore, suppose that Θ *is the integral operator defined in* (3.4) *with a kernel* θ *as in* (3.1), (3.2), (6.208)*. Finally, fix* $\kappa > 0$ *and, with* α_ρ *as in* (2.11) *and* α *as in* (6.208)*, set*

$$\gamma := \min\big\{\alpha_\rho, \alpha\big\}. \tag{6.209}$$

Given $q \in (1,\infty)$ *and* $p \in \big(\frac{d}{d+\gamma},\infty\big)$ *consider the estimate*

$$\Bigg\|\Big(\int_{\Gamma_\kappa(x)} |(\Theta f)(y)|^q \,\frac{d\mu(y)}{\delta_E(y)^{m-q\upsilon}}\Big)^{\frac{1}{q}}\Bigg\|_{L^p_x(E,\sigma)} \leq C\|f\|_{H^p(E,\rho|_E,\sigma)}, \quad \forall\, f \in H^p(E,\rho|_E,\sigma), \tag{6.210}$$

where $C > 0$ *is a finite constant.*

(I) Assume that $q \in (1,\infty)$ *has the property that, for some finite constant* $C > 0$*, either*

$$\Bigg\|\Big(\int_{\Gamma_\kappa(x)} |(\Theta f)(y)|^q \,\frac{d\mu(y)}{\delta_E(y)^{m-q\upsilon}}\Big)^{\frac{1}{q}}\Bigg\|_{L^q_x(E,\sigma)} \leq C\|f\|_{L^q(E,\sigma)}, \quad \forall\, f \in L^q(E,\sigma), \tag{6.211}$$

or there exists $p_o \in (q,\infty)$ *such that for every* $f \in L^{p_o}(E,\sigma)$ *there holds*

$$\sup_{\lambda>0}\Bigg[\lambda\cdot\sigma\Big(\Big\{x\in E : \int_{\Gamma_\kappa(x)} |(\Theta f)(y)|^q \,\frac{d\mu(y)}{\delta_E(y)^{m-q\upsilon}} > \lambda^q\Big\}\Big)^{1/p_o}\Bigg] \leq C\|f\|_{L^{p_o}(E,\sigma)}. \tag{6.212}$$

Then (6.210) *holds for every* $p \in \big(\frac{d}{d+\gamma},\infty\big)$*.*

(II) Assume that $q \in (1,\infty)$ *is such that there exist* $p_o \in (1,\infty)$ *and a finite constant* $C > 0$ *such that* (6.212) *holds for every* $f \in L^{p_o}(E,\sigma)$*. Then* (6.210) *holds for every* $p \in (1,p_o)$ *and, in addition, for every* $f \in L^1(E,\sigma)$ *one has*

$$\sup_{\lambda>0}\Bigg[\lambda\cdot\sigma\Big(\Big\{x\in E : \int_{\Gamma_\kappa(x)} |(\Theta f)(y)|^q \,\frac{d\mu(y)}{\delta_E(y)^{m-q\upsilon}} > \lambda^q\Big\}\Big)\Bigg] \leq C\|f\|_{L^1(E,\sigma)}. \tag{6.213}$$

It is worth mentioning that the conclusion (6.210) in Theorem 6.18 may be re-phrased as saying that the operator

$$\delta_E^{\upsilon-m/q}\Theta : H^p(E,\rho|_E,\sigma) \longrightarrow L^{(p,q)}(\mathscr{X},E) \tag{6.214}$$

is well-defined, linear and bounded.

To set the stage for presenting the proof of Theorem 6.18, we state a lemma containing an estimate for a Marcinkiewicz-type integral (cf. [61] for a proof).

Lemma 6.19. *Assume that (E,ρ,σ) is a d-dimensional* ADR *space for some $d>0$. Then for each $\alpha>0$ there exists $C\in(0,\infty)$ such that whenever F is a nonempty closed subset of (E,τ_ρ) one has*

$$\int_F\int_E \frac{\mathrm{dist}_{\rho_\#}(y,F)^\alpha}{\rho_\#(x,y)^{d+\alpha}}\,d\sigma(y)\,d\sigma(x) \leq C\sigma(E\setminus F). \tag{6.215}$$

We are now prepared to present the

Proof of Theorem 6.18. We divide the proof into a number of cases.

`Case 1`: *Let $q\in[1,\infty)$, $p_o\in(1,\infty)$ be such that* (6.212) *holds for each $f\in L^{p_o}(E,\sigma)$.* The main step in this scenario is proving that the operator $\mathcal{A}_{q,\kappa}\circ(\delta_E^{\upsilon-m/q}\Theta)$ is of weak type $(1,1)$, that is, that there exists $C>0$ such that for every $\lambda>0$ there holds

$$\sigma\Big(\big\{x\in E:\,\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta f)\big)(x)>\lambda\big\}\Big) \leq C\frac{\|f\|_{L^1(E,\sigma)}}{\lambda}, \qquad \forall\, f\in L^1(E,\sigma). \tag{6.216}$$

Assuming (6.216) for the moment, we proceed as follows. The operator $\mathcal{A}_{q,\kappa}\circ(\delta_E^{\upsilon-m/q}\Theta)$ is subadditive, of weak type $(1,1)$ by (6.216), and of weak type (p_o,p_o) by (6.212). Hence, by the Marcinkiewicz interpolation theorem, $\mathcal{A}_{q,\kappa}\circ(\delta_E^{\upsilon-m/q}\Theta)$ is of strong type (p,p) for every $p\in(1,p_o)$, yielding (6.210) (after unraveling notation), for the specified range of q,p_o,p. As such, this takes care of the claim made in the first part of (II) in the statement of the theorem. Moreover, (6.213) corresponds to (6.216), whose proof we now consider.

To get started, assume that $f\in L^1(E,\sigma)$ has been fixed. When $0<\lambda\leq\|f\|_{L^1(E,\sigma)}/\sigma(E)$ (which may only happen in the case when E is bounded), we have

$$\sigma\Big(\big\{x\in E:\,\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta f)\big)(x)>\lambda\big\}\Big) \leq \sigma(E) \leq \frac{\|f\|_{L^1(E,\sigma)}}{\lambda}, \tag{6.217}$$

so (6.216) holds in this case if we choose $C\geq 1$.

Consider now the case when $\lambda>\|f\|_{L^1(E,\sigma)}/\sigma(E)$. There is no loss of generality in assuming that f has bounded support, and we shall perform a Calderón-Zygmund decomposition of f at level λ. More precisely, there exist two finite constants $C>0$, $N\in\mathbb{N}$ (depending only on geometry), along with an at most countable family of balls $(Q_j)_{j\in J}$, say $Q_j:=B_\rho(x_j,r_j)$ for

each $j \in J$, and two functions $g, b : E \to \mathbb{R}$ satisfying the following properties (cf., e.g., [18]):

$$f = g + b \text{ on } E, \tag{6.218}$$

$$g \in L^1(E,\sigma) \cap L^\infty(E,\sigma), \quad \|g\|_{L^1(E,\sigma)} \leq C\|f\|_{L^1(E,\sigma)}, \quad |g(x)| \leq C\lambda, \ \forall\, x \in E, \tag{6.219}$$

$$b = \sum_{j \in J} b_j \ \text{ with } \ \operatorname{supp} b_j \subseteq Q_j, \ \int_E b_j\, d\sigma = 0, \ \text{ and } \ ⨍_{Q_j} |b_j|\, d\sigma \leq C\lambda, \ \ \forall\, j \in J, \tag{6.220}$$

$$\begin{aligned} &\text{if } \mathcal{O} := \textstyle\bigcup_{j \in \mathbb{N}} Q_j \subseteq E \ \text{ and } \ F := E \setminus \mathcal{O}, \ \text{ then } \ \textstyle\sum_{j \in J} \mathbf{1}_{Q_j} \leq N, \\ &\sigma(\mathcal{O}) \leq \tfrac{C}{\lambda}\|f\|_{L^1(E,\sigma)} \ \text{ and } \ \operatorname{dist}_\rho(Q_j, F) \approx r_j \ \text{ uniformly in } j \in J. \end{aligned} \tag{6.221}$$

Note that the above properties also entail $\sum_{j \in J} \|b_j\|_{L^1(E,\sigma)} \leq C\|f\|_{L^1(E,\sigma)}$, so the series in (6.220) converges absolutely in $L^1(E,\sigma)$.

By the quasi-subadditivity of $\mathcal{A}_{q,\kappa} \circ (\delta_E^{\upsilon - m/q}\Theta)$ and (6.218) we have

$$\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon - m/q}(\Theta f)\big) \leq \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon - m/q}(\Theta g)\big) + \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon - m/q}(\Theta b)\big) \tag{6.222}$$

so, as far as (6.216) is concerned, it suffices to prove that

$$\sigma\Big(\big\{x \in E : \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon - m/q}(\Theta g)\big)(x) > \lambda/2\big\}\Big) \leq C\frac{\|f\|_{L^1(E,\sigma)}}{\lambda}, \tag{6.223}$$

and

$$\sigma\Big(\big\{x \in E : \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon - m/q}(\Theta b)\big)(x) > \lambda/2\big\}\Big) \leq C\frac{\|f\|_{L^1(E,\sigma)}}{\lambda}. \tag{6.224}$$

Making use of (6.212) (with f replaced by g), (6.219) and keeping in mind that $p_o > 1$ we obtain

$$\begin{aligned} \sigma\Big(\big\{x \in E : \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon - m/q}(\Theta g)\big)(x) > \lambda/2\big\}\Big) &\leq C\Big(\frac{\|g\|_{L^{p_o}(E,\sigma)}}{\lambda}\Big)^{p_o} \\ &\leq C\frac{\|g\|^{p_o - 1}_{L^\infty(E,\sigma)}\|g\|_{L^1(E,\sigma)}}{\lambda^{p_o}} \leq C\frac{\|f\|_{L^1(E,\sigma)}}{\lambda}, \end{aligned} \tag{6.225}$$

thus (6.223) is proved. We are therefore left with proving (6.224). To justify this, first note that by (6.221) we have

$$\sigma\Big(\big\{x \in \mathcal{O} : \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon - m/q}(\Theta b)\big)(x) > \lambda/2\big\}\Big) \leq \sigma(\mathcal{O}) \leq C\frac{\|f\|_{L^1(E,\sigma)}}{\lambda}. \tag{6.226}$$

Second, it is immediate that

$$\sigma\Big(\big\{x \in F : \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon - m/q}(\Theta b)\big)(x) > \lambda/2\big\}\Big) \leq \frac{1}{\lambda}\int_F \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon - m/q}(\Theta b)\big)\, d\sigma. \tag{6.227}$$

Therefore, since $E = \mathcal{O} \cup F$, in view of (6.226) and (6.227), estimate (6.224) will follow as soon as we prove that

$$\int_F \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon - m/q}(\Theta b)\big)\, d\sigma \leq C\|f\|_{L^1(E,\sigma)} \tag{6.228}$$

for some $C > 0$ independent of f. With this goal in mind, we fix $j \in J$ and $x \in F$ arbitrary and look for a pointwise estimate for

$$\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta b_j)\big)(x) = \Big(\int_{\Gamma_\kappa(x)} |(\Theta b_j)(y)|^q \delta_E(y)^{q\upsilon-m}\, d\mu(y)\Big)^{\frac{1}{q}}. \tag{6.229}$$

With x_j and r_j denoting, respectively, the center and radius of Q_j, based on the third condition in (6.220), for each $y \in \Gamma_\kappa(x)$, we may write

$$\begin{aligned} \big|(\Theta b_j)(y)\big| &= \Big|\int_E \theta(y,z) b_j(z)\, d\sigma(z)\Big| = \Big|\int_E \big[\theta(y,z) - \theta(y,x_j)\big] b_j(z)\, d\sigma(z)\Big| \\ &= \Big|\int_{Q_j} \big[\theta(y,z) - \theta(y,x_j)\big] b_j(z)\, d\sigma(z)\Big| \leq I_1 + I_2, \end{aligned} \tag{6.230}$$

where, for some small $\epsilon > 0$ to be determined momentarily, we have set

$$I_1 := \int\limits_{\substack{z\in Q_j \\ \rho_\#(z,x_j)<\epsilon\rho_\#(y,x_j)}} \big|\theta(y,z) - \theta(y,x_j)\big| |b_j(z)|\, d\sigma(z), \tag{6.231}$$

$$I_2 := \int\limits_{\substack{z\in Q_j \\ \rho_\#(z,x_j)\geq\epsilon\rho_\#(y,x_j)}} \big|\theta(y,z) - \theta(y,x_j)\big| |b_j(z)|\, d\sigma(z). \tag{6.232}$$

Note that, by (2.14), if $\rho_\#(z,x_j) < \epsilon\rho_\#(y,x_j)$ then

$$\rho(z,x_j) \leq C_\rho^2 \rho_\#(z,x_j) < \epsilon C_\rho^2 \rho_\#(y,x_j) \leq \epsilon \widetilde{C}_\rho C_\rho^2 \rho_\#(y,x_j) < \tfrac{1}{2}\rho(y,x_j) \tag{6.233}$$

if $0 < \epsilon < 2^{-1}\widetilde{C}_\rho^{-1} C_\rho^{-2}$. Hence, for this choice of ϵ, we have $\rho(z,x_j) < \frac{1}{2}\rho(y,x_j)$ on the domain of integration in I_1. Based on this, (3.3) and (6.220), we may then estimate this term as follows

$$\begin{aligned} I_1 &\leq C \int_{Q_j} \frac{\rho_\#(z,x_j)^\alpha \delta_E(y)^{-a}}{\rho_\#(y,x_j)^{d+\upsilon+\alpha-a}} |b_j(z)|\, d\sigma(z) \\ &\leq C \frac{r_j^\alpha \delta_E(y)^{-a}}{\rho_\#(y,x_j)^{d+\upsilon+\alpha-a}} \int_{Q_j} |b_j(z)|\, d\sigma(z) \leq C\lambda \frac{r_j^\alpha \delta_E(y)^{-a} \sigma(Q_j)}{\rho_\#(y,x_j)^{d+\upsilon+\alpha-a}}. \end{aligned} \tag{6.234}$$

Estimating I_2 requires a few geometrical preliminaries. Recall that $x \in F$, $y \in \Gamma_\kappa(x)$ and fix an arbitrary point $z \in Q_j$ such that $\rho(z,x_j) \geq \epsilon\rho(y,x_j)$, where $\epsilon > 0$ is as above. Since, on the one hand,

$$\begin{aligned} r_j &\approx \operatorname{dist}_\rho(Q_j,F) \leq \widetilde{C}_\rho \rho(x,x_j) \leq C\rho(x,y) + C\rho(y,x_j) \\ &\leq C(1+\kappa)\delta_E(y) + C\rho(y,x_j) \leq C\rho(y,x_j), \end{aligned} \tag{6.235}$$

while, on the other hand, the fact that $z \in Q_j$ forces $\rho(x_j,z) < r_j$ which in turn allow us to estimate $\rho(y,x_j) \leq \epsilon^{-1}\rho(z,x_j) \leq \epsilon^{-1}\widetilde{C}_\rho \rho(x_j,z) < \epsilon^{-1}\widetilde{C}_\rho r_j$. Hence, ultimately,

$$r_j \approx \rho(y,x_j), \quad \text{uniformly in } j \in J \text{ and } y \in \Gamma_\kappa(x) \text{ with } x \in F. \tag{6.236}$$

In addition, the same type of estimate as in (6.235) written with x_j replaced by z yields $r_j \leq C\rho(y,z)$, which further implies

$$\rho(y,x_j) \leq C\rho(y,z) + C\rho(z,x_j) \leq C\rho(y,z) + Cr_j \leq C\rho(y,z), \tag{6.237}$$

for some constant $C \in (0,\infty)$ independent of j,x,y,z. Hence, from (6.236) and (6.237), we obtain

$$\frac{1}{\rho(y,z)^{d+\upsilon}} \leq \frac{C}{\rho(y,x_j)^{d+\upsilon}} \leq \frac{Cr_j^\alpha}{\rho(y,x_j)^{d+\upsilon+\alpha}}. \tag{6.238}$$

Consequently, on the domain of integration in I_2 we have thanks to (3.2) and (6.238)

$$\big|\theta(y,z) - \theta(y,x_j)\big| \leq \frac{C\delta_E(y)^{-a}}{\rho_\#(y,z)^{d+\upsilon-a}} + \frac{C\delta_E(y)^{-a}}{\rho_\#(y,x_j)^{d+\upsilon-a}} \leq \frac{Cr_j^\alpha \delta_E(y)^{-a}}{\rho_\#(y,x_j)^{d+\upsilon+\alpha-a}}. \tag{6.239}$$

Together with (6.220), this allows us to estimate (recall that I_2 has been defined in (6.232))

$$I_2 \;\leq\; \frac{Cr_j^\alpha \delta_E(y)^{-a}}{\rho_\#(y,x_j)^{d+\upsilon+\alpha-a}} \int\limits_{\substack{z\in Q_j \text{ such that} \\ \rho_\#(z,x_j)\geq \epsilon\rho_\#(y,x_j)}} |b_j(z)|\, d\sigma(z) \leq C\lambda \frac{r_j^\alpha \delta_E(y)^{-a}\sigma(Q_j)}{\rho_\#(y,x_j)^{d+\upsilon+\alpha-a}}. \tag{6.240}$$

Cumulatively, (6.230), (6.234) and (6.240) prove that there exists $C \in (0,\infty)$ with the property that, for every $j \in J$,

$$x \in F \implies \big|(\Theta b_j)(y)\big| \leq C\lambda \frac{r_j^\alpha \delta_E(y)^{-a}\sigma(Q_j)}{\rho_\#(y,x_j)^{d+\upsilon+\alpha-a}}, \qquad \forall\, y \in \Gamma_\kappa(x). \tag{6.241}$$

Utilizing (6.241) in (6.229), it follows that for every $j \in J$ and $x \in F$

$$\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta b_j)\big)(x) \leq C\lambda\, r_j^\alpha \sigma(Q_j)\Big(\int_{\Gamma_\kappa(x)} \frac{\delta_E(y)^{q(\upsilon-a)-m}}{\rho_\#(y,x_j)^{q(d+\upsilon+\alpha-a)}}\, d\mu(y)\Big)^{\frac{1}{q}}. \tag{6.242}$$

At this point we make use of Lemma 6.16 (recall that $\nu - a > 0$) to further bound the last integral in (6.242) and obtain that for every $j \in J$ and $x \in F$

$$\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta b_j)\big)(x) \leq C\lambda\, r_j^\alpha \sigma(Q_j)\rho(x,x_j)^{-d-\alpha} \leq C\lambda \int_{Q_j} \frac{\mathrm{dist}_{\rho_\#}(z,F)^\alpha}{\rho_\#(x,z)^{d+\alpha}}\, d\sigma(z). \tag{6.243}$$

Two geometrical inequalities that have been used in the last step in (6.243) are as follows. First, $\mathrm{dist}_{\rho_\#}(z,F) \approx r_j$, uniformly for $z \in Q_j$ and, second, for every $z \in Q_j$ we have

$$\begin{aligned} \rho(x,z) \;&\leq\; C\rho(x,x_j) + C\rho(x_j,z) \leq C\rho(x,x_j) + Cr_j \\ &\leq\; C\rho(x,x_j) + C\,\mathrm{dist}_\rho(Q_j,F) \leq C\rho(x,x_j). \end{aligned} \tag{6.244}$$

Summing up inequalities of the form (6.243) over $j \in \mathbb{N}$ and using the sublinearity of the operator $\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}\Theta(\cdot)\big)$ (recall that $q \geq 1$), as well as the finite overlap property in (6.221), we obtain

$$\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta b)\big)(x) \leq C\lambda \int_{\mathcal{O}} \frac{\mathrm{dist}_{\rho_\#}(z,F)^\alpha}{\rho_\#(x,z)^{d+\alpha}}\, d\sigma(z), \qquad \forall\, x \in F. \tag{6.245}$$

Consequently, from (6.245), Lemma 6.19 and (6.221), we deduce that

$$\begin{aligned}\int_F \mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta b)\big)(x)\,dx &\leq C\lambda\int_F\int_{\mathcal{O}}\frac{\operatorname{dist}_{\rho_\#}(z,F)^\alpha}{\rho_\#(x,z)^{d+\alpha}}\,d\sigma(z)\,d\sigma(x) \\ &\leq C\lambda\int_F\int_E\frac{\operatorname{dist}_{\rho_\#}(z,F)^\alpha}{\rho_\#(x,z)^{d+\alpha}}\,d\sigma(z)\,d\sigma(x)\leq C\lambda\,\sigma(E\setminus F)=C\lambda\,\sigma(\mathcal{O})\leq C\|f\|_{L^1(E,\sigma)}.\end{aligned}\tag{6.246}$$

This proves (6.228), thus completing the proof of (6.216). In summary, the analysis so far proves part (II) in the statement of the theorem.

`Case 2`: *Assume that* (6.212) *holds for some* $1<q<p_o<\infty$. As a preliminary step, we make the observation that, in this scenario, granted (6.212) and the conclusion in the Case 1,

$$\mathscr{A}_{q,\kappa}\circ\big(\delta_E^{\upsilon-m/q}\Theta\big):L^r(E,\sigma)\to L^r(E,\sigma)\quad\text{is bounded whenever } r\in(1,p_o).\tag{6.247}$$

Because of the equivalence (6.102) in Theorem 6.10, estimate (6.210) for the range $p\in(q,\infty)$ will follow once we show that

$$\mathfrak{C}_{q,\kappa}\circ(\delta_E^{\upsilon-m/q}\Theta):L^p(E,\sigma)\to L^p(E,\sigma)\quad\text{is bounded for } q<p<\infty.\tag{6.248}$$

Fix $p\in(q,\infty)$. The proof of the boundedness of the operator in (6.248) relies on the following pointwise estimate

$$\begin{aligned}&\mathfrak{C}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta f)\big)(x_0) \\ &\qquad\leq C\big[\big(M_E(|f|^q)(x_0)\big)^{\frac{1}{q}}+(M_E(M_E(f)))(x_0)\big],\qquad\forall\,x_0\in E,\end{aligned}\tag{6.249}$$

for each $f\in L^p(E,\sigma)$. Indeed, fix such a function f. After raising the inequality in (6.249) to the p-th power and then integrating over E, we obtain

$$\begin{aligned}&\int_E\big[\mathfrak{C}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}(\Theta f)\big)(x)\big]^p\,d\sigma(x) \\ &\qquad\leq C\int_E\big[M_E(|f|^q)(x)\big]^{\frac{p}{q}}\,d\sigma(x)+C\int_E\big[(M_E^2f)(x)\big]^p\,d\sigma(x)\leq C\int_E|f|^p\,d\sigma,\end{aligned}\tag{6.250}$$

where the last inequality in (6.250) uses the boundedness on $L^p(E,\sigma)$ and $L^{p/q}(E,\sigma)$ of the Hardy-Littlewood maximal operator M_E (here we make use of the fact that in the current case $p>\max\{q,1\}$). This shows that (6.248) holds assuming (6.249).

Returning to the proof of (6.249), fix $f\in L^p(E,\sigma)$ along with $r>0$ and $x_0\in E$. For some finite constant $c>0$ to be specified later, set $\Delta:=E\cap B_{\rho_\#}(x_0,r)$ and $c\Delta:=E\cap B_{\rho_\#}(x_0,cr)$, then write $f=f_1+f_2$, where $f_1:=f\mathbf{1}_{c\Delta}$ and $f_2:=f\mathbf{1}_{E\setminus c\Delta}$. First we estimate the contribution

from f_1 by writing

$$\begin{aligned}
&\tfrac{1}{\sigma(\Delta)}\int_{\mathcal{T}_\kappa(\Delta)}|(\Theta f_1)(x)|^q\delta_E(x)^{qv-(m-d)}\,d\mu(x)\\
&\qquad\le \tfrac{1}{\sigma(\Delta)}\int_{\mathscr{X}\setminus E}|(\Theta f_1)(x)|^q\delta_E(x)^{qv-(m-d)}\,d\mu(x)\\
&\qquad\le \tfrac{C}{\sigma(\Delta)}\int_E\Big(\int_{\Gamma_\kappa(x)}|(\Theta f_1)(y)|^q\,\delta_E(y)^{qv-m}\,d\mu(y)\Big)\,d\sigma(x)\\
&\qquad= C\big\|\mathscr{A}_{q,\kappa}\big(\delta_E^{v-m/q}(\Theta f_1)\big)\big\|^q_{L^q(E,\sigma)}\le \tfrac{C}{\sigma(\Delta)}\int_E|f_1|^q\,d\sigma\\
&\qquad= \tfrac{C}{\sigma(\Delta)}\int_{c\,\Delta}|f|^q\,d\sigma\le C\,M_E(|f|^q)(x_0).
\end{aligned}\tag{6.251}$$

For the second inequality in (6.251) we have used (6.58), Lemma 6.3, and the fact that $(E,\rho|_E,\sigma)$ is a d-dimensional ADR space, while the third inequality follows from (6.247) used with $r:=q\in(1,p_o)$.

To treat the term corresponding to f_2, observe that if $c>C_\rho$, then for every $y\in E\setminus c\,\Delta$ we have $cr<\rho_\#(y,x_0)\le C_\rho\max\{\rho_\#(y,w),\rho_\#(w,x_0)\}\le C_\rho\rho_\#(y,w)$ for every $w\in\Delta$. Hence, $E\setminus c\,\Delta\subseteq\{y\in E:\ \rho_\#(y,w)>r\}$ and $\rho_\#(y,x_0)\approx\rho_\#(y,w)$, uniformly for $y\in E\setminus c\,\Delta$ and $w\in\Delta$. Furthermore, for every $z\in\mathcal{T}_\kappa(\Delta)$, we have

$$\begin{aligned}
\rho_\#(y,x_0)&\le C_\rho\max\{\rho_\#(y,z),\rho_\#(z,x_0)\}\le C_\rho\max\{\rho_\#(y,z),(1+\kappa)\delta_E(z)\}\\
&\le C\rho_\#(y,z).
\end{aligned}\tag{6.252}$$

Based on these considerations as well as (3.2) and (3.20), if $z\in\mathcal{T}_\kappa(\Delta)$ we may write

$$\begin{aligned}
|(\Theta f_2)(z)|&\le C\int_{E\setminus c\,\Delta}\frac{\delta_E(z)^{-a}}{\rho_\#(z,y)^{d+v-a}}|f(y)|\,d\sigma(y)\\
&\le \frac{C\delta_E(z)^{-a}}{r^{v-a}}\int_{y\in E,\,\rho_\#(y,w)>r}\frac{r^{v-a}}{\rho(y,w)^{d+v-a}}|f(y)|\,d\sigma(y)\\
&\le \frac{C\delta_E(z)^{-a}}{r^{v-a}}(M_Ef)(w),\quad\text{uniformly for } w\in\Delta.
\end{aligned}\tag{6.253}$$

Thus, (6.253) implies

$$|(\Theta f_2)(z)|\le\frac{C\delta_E(z)^{-a}}{r^{v-a}}\inf_{w\in\Delta}(M_Ef)(w),\qquad\forall\,z\in\mathcal{T}_\kappa(\Delta).\tag{6.254}$$

In concert with (6.25) and Lemma 3.6 (which uses $v-a>0$), estimate (6.254) further yields

$$\begin{aligned}
&\Big[\tfrac{1}{\sigma(\Delta)}\int_{\mathcal{T}_\kappa(\Delta)}|(\Theta f_2)(z)|^q\delta_E(z)^{qv-(m-d)}\,d\mu(z)\Big]^{\frac1q}\\
&\qquad\le\frac{C}{r^{v-a}}\inf_{w\in\Delta}(M_Ef)(w)\Big[\tfrac{1}{\sigma(\Delta)}\int_{B_{\rho_\#}(x_0,Cr)\setminus E}\delta_E(z)^{q(v-a)-(m-d)}\,d\mu(z)\Big]^{\frac1q}\\
&\qquad\le C\inf_{w\in\Delta}(M_Ef)(w)\le C{\int\!\!\!\!\!\!-}_{\Delta}M_Ef\,d\sigma\le CM_E(M_Ef)(x_0).
\end{aligned}\tag{6.255}$$

Now (6.249) follows from (6.251) and (6.253) in view of (6.97) and the fact that $\mathfrak{C}_{q,\kappa}\circ\big(\delta_E^{\upsilon-m/q}\Theta\big)$ is, in the current case, sub-linear.

In summary, the analysis in this case proves that, under the assumption (6.212), estimate (6.210) holds whenever $1<q<p_o<\infty$ and $q<p<\infty$.

`Case 3`: *Assume that* $q\in(1,\infty)$ *is such that* (6.211) *holds.* We claim that

$$\mathfrak{C}_{q,\kappa}\circ(\delta_E^{\upsilon-m/q}\Theta):L^p(E,\sigma)\to L^p(E,\sigma)\quad\text{is bounded for } q<p<\infty. \tag{6.256}$$

The proof of (6.256) largely parallels that of (6.248). More specifically, the only significant difference occurs in the third inequality in (6.251) which, this time, follows directly from (6.211). Once this has been established, the equivalence (6.102) in Theorem 6.10, and the current assumption yield (6.210) for the range $p\in[q,\infty)$.

`Case 4`: *Assume* $\frac{d}{d+\gamma}<p\leq 1$ *and* $q\in[p,\infty)$*, and suppose that*

$$\mathscr{A}_{q,\kappa}\circ\big(\delta_E^{\upsilon-m/q}\Theta\big):L^q(E,\sigma)\to L^q(E,\sigma)\quad\text{is bounded.} \tag{6.257}$$

In this case, we shall prove that there exists $C\in(0,\infty)$ such that

$$\big\|\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}\Theta(a)\big)\big\|^p_{L^p(E,\sigma)}\leq C,\qquad\text{for every } p\text{-atom } a. \tag{6.258}$$

With this goal in mind, fix a p-atom a and let $x_0\in E$ and $r>0$ be such that the conditions in (6.199) hold. In particular,

$$\operatorname{supp}a\subseteq B_{\rho_\#}(x_0,\widetilde{C}_\rho r). \tag{6.259}$$

Then, for some finite constant $c>1$ to be specified later, and with $\Delta:=E\cap B_{\rho_\#}(x_0,cr)$, we have

$$\begin{aligned}\big\|\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}\Theta(a)\big)\big\|^p_{L^p(E,\sigma)}&=\int_\Delta\Big(\int_{\Gamma_\kappa(x)}|(\Theta a)(y)|^q\delta_E(y)^{q\upsilon-m}\,d\mu(y)\Big)^{\frac{p}{q}}d\sigma(x)\\&\quad+\int_{E\setminus\Delta}\Big(\int_{\Gamma_\kappa(x)}|(\Theta a)(y)|^q\delta_E(y)^{q\upsilon-m}\,d\mu(y)\Big)^{\frac{p}{q}}d\sigma(x)=:I_1+I_2.\end{aligned} \tag{6.260}$$

Using Hölder's inequality (with exponent $q/p\geq 1$), the fact that $(E,\rho|_E,\sigma)$ is a d-dimensional ADR space, (6.257), and (6.199), we may write

$$\begin{aligned}I_1&\leq C\left[\int_\Delta\Big(\int_{\Gamma_\kappa(x)}|(\Theta a)(y)|^q\delta_E(y)^{q\upsilon-m}\,d\mu(y)\Big)\,d\sigma(x)\right]^{\frac{p}{q}}r^{d\left(1-\frac{p}{q}\right)}\\&\leq C\big\|\mathcal{A}_{q,\kappa}\big(\delta_E^{\upsilon-m/q}\Theta(a)\big)\big\|^p_{L^q(E,\sigma)}r^{d\left(1-\frac{p}{q}\right)}\leq C\|a\|^p_{L^q(E,\sigma)}r^{d\left(1-\frac{p}{q}\right)}\leq C,\end{aligned} \tag{6.261}$$

for some finite $C>0$ independent of a. We are left with estimating I_2. First, we look for a pointwise estimate for Θa. Fix $x\in E\setminus\Delta$ and $y\in\Gamma_\kappa(x)$. Then for every $z\in E\cap B_{\rho_\#}(x_0,\widetilde{C}_\rho r)$ we have

$$\begin{aligned}\rho_\#(x_0,z)&\leq\widetilde{C}_\rho r\leq\tfrac{1}{c}\,\rho_\#(x,x_0)\leq\tfrac{1}{c}\,\widetilde{C}_\rho C_{\rho_\#}\max\{\rho_\#(x,y),\rho_\#(y,x_0)\}\\&\leq\tfrac{1}{c}\,\widetilde{C}_\rho C_\rho\max\{(1+\kappa)\delta_E(y),\rho_\#(y,x_0)\}\\&\leq\tfrac{1}{c}\,\widetilde{C}_\rho C_\rho(1+\kappa)\rho_\#(y,x_0).\end{aligned} \tag{6.262}$$

Now, based on this and (2.14), by choosing c sufficiently large we conclude that

$$\rho(z,x_0)\leq \tfrac{1}{2}\rho(y,x_0)\quad \text{for every}\quad z\in E\cap B_{\rho_\#}(x_0,\widetilde{C}_\rho r). \tag{6.263}$$

At this point, we may use the last condition in (6.199), (6.259), (6.263), (6.208), the second condition in (6.199) and the fact that $(E,\rho|_E,\sigma)$ is a d-dimensional ADR space in order to obtain

$$\begin{aligned}
|(\Theta a)(y)| &= \Big|\int_E [\theta(y,z)-\theta(y,x_0)]a(z)\,d\sigma(z)\Big| \\
&= \Big|\int_{E\cap B_{\rho_\#}(x_0,\widetilde{C}_\rho r)} [\theta(y,z)-\theta(y,x_0)]a(z)\,d\sigma(z)\Big| \\
&\leq C\int_{E\cap B_{\rho_\#}(x_0,\widetilde{C}_\rho r)} \frac{\rho_\#(z,x_0)^\alpha\delta_E(y)^{-a}}{\rho_\#(y,x_0)^{d+\upsilon+\alpha-a}}\,|a(z)|\,d\sigma(z) \\
&\leq C2^{\gamma-\alpha}\delta_E(y)^{-a}\int_{E\cap B_{\rho_\#}(x_0,\widetilde{C}_\rho r)} \frac{\rho_\#(z,x_0)^\gamma}{\rho_\#(y,x_0)^{d+\upsilon-a+\gamma}}\,|a(z)|\,d\sigma(z) \\
&\leq C\frac{\delta_E(y)^{-a}r^{\gamma+d\left(1-\frac{1}{p}\right)}}{\rho_\#(y,x_0)^{d+\upsilon-a+\gamma}},\qquad \forall\, y\in\Gamma_\kappa(x).
\end{aligned} \tag{6.264}$$

In turn, (6.264) yields

$$\begin{aligned}
\int_{\Gamma_\kappa(x)}|(\Theta a)(y)|^q\frac{d\mu(y)}{\delta_E(y)^{m-q\upsilon}} &\leq C r^{q\gamma+qd\left(1-\frac{1}{p}\right)}\int_{\Gamma_\kappa(x)}\frac{\delta_E(y)^{q(\upsilon-a)-m}}{\rho_\#(y,x_0)^{q(d+\upsilon-a+\gamma)}}\,d\mu(y) \\
&\leq C\frac{r^{q\gamma+qd\left(1-\frac{1}{p}\right)}}{\rho_\#(x,x_0)^{qd+q\gamma}},\qquad \forall\, x\in E\setminus\Delta,
\end{aligned} \tag{6.265}$$

where for the last inequality in (6.265) we applied Lemma 6.16. Estimate (6.265) used in I_2 further implies

$$\begin{aligned}
I_2 &\leq C\,r^{p\gamma+pd\left(1-\frac{1}{p}\right)}\int_{E\setminus\Delta}\frac{d\sigma(x)}{\rho_\#(x,x_0)^{pd+p\gamma}} \\
&\leq C\frac{r^{p\gamma+pd\left(1-\frac{1}{p}\right)}}{r^{pd+p\gamma-d}}=C,
\end{aligned} \tag{6.266}$$

where the last inequality in (6.266) is a consequence of (3.20) (used with $f\equiv 1$) and the fact that $p(d+\gamma)>d$. Now (6.258) follows from (6.260), (6.261) and (6.266).

`Case 5`: *Assume* $\frac{d}{d+\gamma}<p\leq 1\leq q<\infty$*, and suppose that* (6.257) *holds.* Then we claim that

$$\begin{gathered}
\delta_E^{\upsilon-m/q}\Theta : H^p(E,\rho|_E,\sigma)\to L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)\quad \text{is bounded} \\
\text{whenever } \tfrac{d}{d+\gamma}<p\leq 1\leq q<+\infty.
\end{gathered} \tag{6.267}$$

To proceed with the proof of this claim, fix p and q as in (6.267) and define the sets

$$\dot{\mathscr{C}}^{\gamma}_{b,0}(E,\rho|_E) := \Big\{ f \in \dot{\mathscr{C}}^{\gamma}(E,\rho|_E) : \, f \text{ has bounded support and } \textstyle\int_E f\,d\sigma = 0 \Big\} \tag{6.268}$$

and

$$\mathcal{F}(E) := \begin{cases} \dot{\mathscr{C}}^{\gamma}_{b,0}(E,\rho|_E) \ \text{ if } E \text{ is unbounded,} \\ \dot{\mathscr{C}}^{\gamma}_{b,0}(E,\rho|_E) \cup \{\mathbf{1}_E\} \ \text{ if } E \text{ is bounded.} \end{cases} \tag{6.269}$$

Then letting

$$\mathcal{D}_0(E) := \text{the finite linear span of functions in } \mathcal{F}(E), \tag{6.270}$$

we shall show that

$$\mathcal{D}_0(E) \ \text{ is dense in } H^p(E,\rho|_E,\sigma). \tag{6.271}$$

Indeed, since finite linear spans of p-atoms are dense in $H^p(E,\rho|_E,\sigma)$, the density result formulated in (6.271) will follow once we show that individual p-atoms may be approximated in $H^p(E,\rho|_E,\sigma)$ with functions from $\mathcal{D}_0(E)$. To prove the latter, recall the approximation to the identity of order γ as given in Proposition 2.14 and observe that from the properties of the integral kernels from Definition 2.13 we have that $\mathcal{S}_l\, a \in \mathcal{D}_0(E)$ for every p-atom a and each $l \in \mathbb{N}$. This and [48, Lemma 3.2, (iii), p. 108], which gives that

$$\begin{aligned} &\{\mathcal{S}_l\}_{l\in\mathbb{N}} \text{ is uniformly bounded from } H^p(E,\rho|_E,\sigma) \text{ to } H^p(E,\rho|_E,\sigma) \ \text{ and} \\ &\mathcal{S}_l f \to f \ \text{ in } H^p(E,\rho|_E,\sigma) \text{ as } l \to +\infty, \quad \forall\, f \in H^p(E,\rho|_E,\sigma), \end{aligned} \tag{6.272}$$

now yield the desired conclusion, finishing the proof of (6.271).

The next task is to prove that there exists $C \in (0,\infty)$ such that

$$\|\delta_E^{\upsilon-m/q}\Theta f\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)} \leq C\|f\|_{H^p(E,\rho|_E,\sigma)}, \quad \forall\, f \in \mathcal{D}_0(E). \tag{6.273}$$

Assume for the moment (6.273). Then, it follows that the linear operator $\delta_E^{\upsilon-m/q}\Theta$ is bounded from $\mathcal{D}_0(E)$ into $L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)$. Based on this, (6.271) and the fact hat the mixed-norm spaces $L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)$ are quasi-Banach (see [62], [9]), it follows that $\delta_E^{\upsilon-m/q}\Theta$ extends in a standard way to a linear operator from $H^p(E,\rho|_E,\sigma)$ into $L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)$. Since the latter spaces are only quasi-normed, to show that this extension is also bounded we use the following property of quasi-normed spaces (for a proof see [60, Theorem 1.5, (6)])

$$\begin{aligned} &\text{if } (X,\|\cdot\|) \text{ is a quasi-normed vector space, then } \exists\, C \in [1,\infty) \text{ such that} \\ &\text{if } x_j \to x_* \text{ in } X \text{ as } j \to \infty, \text{ in the topology induced on } X \text{ by } \|\cdot\|, \text{ then} \\ &\qquad C^{-1}\|x_*\| \leq \liminf_{j\to\infty}\|x_j\| \leq \limsup_{j\to\infty}\|x_j\| \leq C\|x_*\|. \end{aligned} \tag{6.274}$$

In summary, the boundedness claimed in (6.267) follows, once (6.273) is proved.

With the goal of establishing (6.273), fix a function $f \in \dot{\mathscr{C}}^{\gamma}_{b,0}(E,\rho|_E)$. By [48, Proposition 3.1, p. 112], we have that

$$\begin{aligned} &\exists\,(\lambda_j)_{j\in\mathbb{N}} \in \ell^p, \quad \exists\,(a_j)_{j\in\mathbb{N}} \ p\text{-atoms, such that } \Big(\sum_{j=1}^{\infty}|\lambda_j|^p\Big)^{1/p} \leq C\|f\|_{H^p(E,\rho|_E,\sigma)} \\ &\qquad \text{and } \ f = \sum_{j=1}^{\infty}\lambda_j a_j \ \text{ both in } H^p(E,\rho|_E,\sigma) \text{ and in } L^q(E,\sigma), \end{aligned} \tag{6.275}$$

for some $C \in (0, \infty)$ independent of f. Also, from our assumption (6.257) we deduce that

$$\delta_E^{\upsilon-m/q}\Theta : L^q(E,\sigma) \to L^{(q,q)}(\mathscr{X}, E, \mu, \sigma; \kappa) \quad \text{is linear and bounded.} \tag{6.276}$$

Combining (6.276) with (6.275) it follows that, with f as above,

$$\begin{aligned}
\delta_E^{\upsilon-m/q}\Theta f &= \delta_E^{\upsilon-m/q}\Theta\Big(\lim_{N\to\infty}\sum_{j=1}^{N}\lambda_j a_j\Big) = \lim_{N\to\infty}\delta_E^{\upsilon-m/q}\Theta\Big(\sum_{j=1}^{N}\lambda_j a_j\Big) \\
&= \lim_{N\to\infty}\sum_{j=1}^{N}\lambda_j\delta_E^{\upsilon-m/q}\Theta a_j \quad \text{in} \quad L^{(q,q)}(\mathscr{X}, E, \mu, \sigma; \kappa).
\end{aligned} \tag{6.277}$$

Granted this, we may apply [62, Theorem 1.5] to conclude that

$$\begin{gathered}
\exists\,(N_k)_{k\in\mathbb{N}},\ N_k \nearrow +\infty \ \text{ as } \ k \to +\infty, \ \text{ such that} \\
\sum_{j=1}^{N_k}\lambda_j\delta_E^{\upsilon-m/q}\Theta a_j \to \delta_E^{\upsilon-m/q}\Theta f \text{ pointwise } \mu\text{-a.e. on } \mathscr{X}\setminus E \text{ as } k \to +\infty.
\end{gathered} \tag{6.278}$$

Since we are currently assuming that $0 < p \le 1 \le q < \infty$, an inspection of definition (6.11) of the quasi-norm for the space $L^{(p,q)}(\mathscr{X}, E, \mu, \sigma; \kappa)$ reveals that $\|\cdot\|^p_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)}$ is subadditive. As such, for each $k \in \mathbb{N}$, we may estimate

$$\begin{aligned}
\Big\|\sum_{j=1}^{N_k}\lambda_j\delta_E^{\upsilon-m/q}\Theta a_j\Big\|^p_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)} &\le \sum_{j=1}^{N_k}\Big\|\lambda_j\delta_E^{\upsilon-m/q}\Theta a_j\Big\|^p_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)} \\
&= \sum_{j=1}^{N_k}|\lambda_j|^p\Big\|\delta_E^{\upsilon-m/q}\Theta a_j\Big\|^p_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)} \\
&\le C\sum_{j=1}^{N_k}|\lambda_j|^p,
\end{aligned} \tag{6.279}$$

where for the last inequality in (6.279) we used estimate (6.258) (note that the assumptions in Case 4 are currently satisfied). Next, introduce

$$F_k := \sum_{j=1}^{N_k}\lambda_j\delta_E^{\upsilon-m/q}\Theta a_j, \qquad \forall\, k \in \mathbb{N}. \tag{6.280}$$

To proceed, observe that Fatou's lemma holds in the space $L^{(p,q)}(\mathscr{X}, E, \mu, \sigma; \kappa)$ (this is seen directly from (6.11) by applying twice the classical Fatou's lemma in Lebesgue spaces). When used for the sequence $\{F_k\}_{k\in\mathbb{N}}$, this yields

$$\begin{aligned}
\|\delta_E^{\upsilon-m/q}\Theta f\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)} &= \big\|\liminf_{k\to\infty}|F_k|\big\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)} \\
&\le \liminf_{k\to\infty}\|F_k\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)} \le C\|f\|_{H^p(E,\rho|_E,\sigma)}.
\end{aligned} \tag{6.281}$$

The equality in (6.281) is a consequence of (6.278) and the fact that

$$\|u\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)} = \|\,|u|\,\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)}, \quad \forall\, u \in L^{(p,q)}(\mathscr{X}, E, \mu, \sigma; \kappa), \tag{6.282}$$

the first inequality is due to Fatou's Lemma and (6.282), while the last inequality follows from (6.279), (6.280) and (6.275).

At this stage, we have established (6.281) for any function $f \in \dot{\mathscr{C}}^{\gamma}_{b,0}(E, \rho|_E)$, so in order to finish the proof of (6.273) there remains to consider the case when E is bounded and $f = \mathbf{1}_E$. In this setting, since E is d-dimensional ADR, we have $\sigma(E) < \infty$, and we may write

$$\begin{aligned} \big\|\delta_E^{\upsilon-m/q}\Theta\mathbf{1}_E\big\|_{L^{(p,q)}(\mathscr{X},E,\mu,\sigma;\kappa)} &= \big\|\mathscr{A}_{q,\kappa}\circ\big(\delta_E^{\upsilon-m/q}\Theta\mathbf{1}_E\big)\big\|_{L^p(E,\sigma)} \\ &\leq \sigma(E)^{\frac{1}{p}-\frac{1}{q}}\big\|\mathscr{A}_{q,\kappa}\circ\big(\delta_E^{\upsilon-m/q}\Theta\mathbf{1}_E\big)\big\|_{L^q(E,\sigma)} \\ &\leq C\sigma(E)^{\frac{1}{p}} = C < +\infty. \end{aligned} \tag{6.283}$$

The first inequality in (6.283) uses Hölder's inequality for the integrability index $q/p \geq 1$, while the second inequality uses (6.257). Now (6.273) follows by combining (6.281) and (6.283), and with it the proof of (6.267) is finished. In particular, (6.273) may be rewritten as

$$\begin{gathered} \mathscr{A}_{q,\kappa}\circ\big(\delta_E^{\upsilon-m/q}\Theta\big) : H^p(E,\rho|_E,\sigma) \to L^p(E,\sigma) \quad \text{is bounded} \\ \text{whenever } \tfrac{d}{d+\gamma} < p \leq 1 \leq q < +\infty, \text{ and (6.257) holds.} \end{gathered} \tag{6.284}$$

The end-game in the proof of part (I) in the statement of the theorem is now as follows. Assume first that $q \in (1,\infty)$, $p_o \in (q,\infty)$ are such that (6.212) holds for every $f \in L^{p_o}(E,\sigma)$. Based on these assumptions and Case 1 we conclude that

$$\mathscr{A}_{q,\kappa}\circ\big(\delta_E^{\upsilon-m/q}\Theta\big) : L^p(E,\sigma) \to L^p(E,\sigma) \quad \text{is bounded whenever } p \in (1,p_o). \tag{6.285}$$

In particular, (6.285) with $p := q \in (1,p_o)$ and (6.284) yield

$$\mathscr{A}_{q,\kappa}\circ\big(\delta_E^{\upsilon-m/q}\Theta\big) : H^p(E,\rho|_E,\sigma) \to L^p(E,\sigma) \quad \text{is bounded if } \tfrac{d}{d+\gamma} < p \leq 1. \tag{6.286}$$

To proceed, fix an exponent

$$p_o' \in (q,p_o). \tag{6.287}$$

Then (6.285) corresponding to $p := p_o'$ together with Case 2 used here with p_o replaced by p_o' imply that

$$\mathscr{A}_{q,\kappa}\circ\big(\delta_E^{\upsilon-m/q}\Theta\big) : L^p(E,\sigma) \to L^p(E,\sigma) \quad \text{is bounded for each } p \in (q,\infty). \tag{6.288}$$

Now the claim in part (I) in the statement of the theorem corresponding to the current working hypotheses follows from (6.285), (6.286), and (6.288).

There remains to consider the situation when $q \in (1,\infty)$ is such that (6.211) holds. From Case 3 we know that (6.210) is valid in the range $p \in [q,\infty)$. Then in combination with Case 1 (used with $p_o := q \in (1,\infty)$), this gives that

$$\mathscr{A}_{q,\kappa}\circ\big(\delta_E^{\upsilon-m/q}\Theta\big) : L^p(E,\sigma) \to L^p(E,\sigma) \quad \text{is bounded for each } p \in (1,q]. \tag{6.289}$$

Now reasoning as before we obtain that (6.286) holds. In summary, the above analysis shows that (6.210) is valid in the range $p \in \big(\frac{d}{d+\gamma},\infty\big)$, under the assumption that $q \in (1,\infty)$ is such that (6.211) holds. This concludes the proof of part (I), and finishes the proof of the theorem. $\square$

The second main result in this subsection is a combination of Theorem 6.12 and Theorem 6.18.

Theorem 6.20. *Suppose that d, m are real numbers such that $0 < d < m$. Assume that $(\mathscr{X}, \rho, \mu)$ is an m-dimensional* ADR *space, E is a closed subset of $(\mathscr{X}, \tau_\rho)$, and σ is a Borel regular measure on $(E, \tau_{\rho|_E})$ with the property that $(E, \rho\big|_E, \sigma)$ is a d-dimensional* ADR *space. In addition, suppose that Θ is the integral operator defined in* (3.4) *with a kernel θ as in* (3.1), (3.2), (6.208). *Finally, fix $\kappa > 0$ and recall the exponent γ from* (6.209).

If there exist $p_o \in (0, \infty)$ and a finite constant $C_o > 0$ such that for every $f \in L^{p_o}(E, \sigma)$

$$\sup_{\lambda>0} \left[\lambda \cdot \sigma\Big(\Big\{ x \in E : \int_{\Gamma_\kappa(x)} |(\Theta f)(y)|^2 \, \frac{d\mu(y)}{\delta_E(y)^{m-2\upsilon}} > \lambda^2 \Big\} \Big)^{1/p_o} \right] \leq C_o \|f\|_{L^{p_o}(E,\sigma)}, \tag{6.290}$$

then for each $p \in \big(\frac{d}{d+\gamma}, \infty\big)$ there holds

$$\left\| \Big(\int_{\Gamma_\kappa(x)} |(\Theta f)(y)|^2 \, \frac{d\mu(y)}{\delta_E(y)^{m-2\upsilon}} \Big)^{\frac{1}{2}} \right\|_{L^p_x(E,\sigma)} \leq C \|f\|_{H^p(E,\rho|_E,\sigma)}, \quad \forall\, f \in H^p(E, \rho|_E, \sigma), \tag{6.291}$$

where $C > 0$ is a finite constant which is allowed to depend only on p, C_o, κ, C_θ, and geometry.

Proof. The assumption that the operator $\mathcal{A}_{2,\kappa} \circ (\delta_E^{\upsilon - m/2}\Theta) : L^{p_o}(E, \sigma) \to L^{p_o,\infty}(E, \sigma)$ is bounded implies that (6.146) holds. Consequently, Theorem 6.12 applies and yields that $\mathcal{A}_{2,\kappa} \circ (\delta_E^{\upsilon - m/2}\Theta) : L^2(E, \sigma) \to L^2(E, \sigma)$ is bounded as well. With this in hand, part (I) in Theorem 6.18 (pertaining to condition (6.211) with $q = 2$) applies and gives that (6.291) holds for every $p \in \big(\frac{d}{d+\gamma}, \infty\big)$. ☐

7 Conclusion

Theorem 1.1 asserts the equivalence of a number of the properties encountered in the body of the manuscript. A formal proof is presented below.

Proof of Theorem 1.1. The fact that *(1)* $\Rightarrow$ *(2)* is a consequence of Theorem 3.2. It is easy to see that if *(2)* holds, then *(7)* holds by taking $b_Q := \mathbf{1}_Q$ for each $Q \in \mathbb{D}(E)$, hence *(2)* $\Rightarrow$ *(7)*. The implication *(7)* $\Rightarrow$ *(1)* is proved in Theorem 3.7. The implication *(9)* $\Rightarrow$ *(1)* is proved in Theorem 4.3. Moreover, *(1)* $\Leftrightarrow$ *(9)* $\Leftrightarrow$ *(10)* by Theorem 4.5. The implication *(11)* $\Rightarrow$ *(12)* is proved in Theorem 6.20. Clearly *(12)* $\Rightarrow$ *(11)*, while *(11)* $\Rightarrow$ *(1)* is contained in Theorem 6.12. To show that *(1)* $\Rightarrow$ *(11)*, suppose *(1)* holds and take $f \in L^2(E, \sigma)$ and $\lambda > 0$ arbitrary. Then

starting with Tschebyshev's inequality we may write

$$\begin{aligned}
\lambda^2\cdot\sigma\Big(\Big\{x\in E:\int_{\Gamma_\kappa(x)}|(\Theta f)(y)|^2\,\frac{d\mu(y)}{\delta_E(y)^{m-2\upsilon}}>\lambda^2\Big\}\Big)&\\
&\leq\int_{\Big\{x\in E:\int_{\Gamma_\kappa(x)}|(\Theta f)(y)|^2\,\frac{d\mu(y)}{\delta_E(y)^{m-2\upsilon}}>\lambda^2\Big\}}\Big(\int_{\Gamma_\kappa(x)}|(\Theta f)(y)|^2\,\frac{d\mu(y)}{\delta_E(y)^{m-2\upsilon}}\Big)d\sigma(x)\\
&\leq\int_E\Big(\int_{\Gamma_\kappa(x)}|(\Theta f)(y)|^2\,\frac{d\mu(y)}{\delta_E(y)^{m-2\upsilon}}\Big)d\sigma(x)\\
&\leq\int_{\mathscr{X}\setminus E}\frac{|(\Theta f)(y)|^2}{\delta_E(y)^{m-2\upsilon}}\,\sigma(\pi_y^\kappa)\,d\mu(y)\\
&\leq\int_{\mathscr{X}\setminus E}\frac{|(\Theta f)(y)|^2}{\delta_E(y)^{m-2\upsilon}}\,\sigma\Big(E\cap B_{\rho_\#}\big(y_*,C_\rho(1+\kappa)\delta_E(y)\big)\Big)\,d\mu(y)\\
&\leq C\int_{\mathscr{X}\setminus E}|(\Theta f)(y)|^2\delta_E(y)^{2\upsilon-(m-d)}\,d\mu(y)\\
&\leq C\|f\|^2_{L^2(E,\sigma)}.
\end{aligned}\tag{7.1}$$

The third inequality in (7.1) is due to (6.58) (recall (6.17)), the fourth uses (6.32) in Lemma 6.3, the fifth uses the fact that $\big(E,\rho\big|_E,\sigma\big)$ is a d-dimensional ADR space, and the last inequality is a consequence of (1.26). Thus, *(1)* $\Rightarrow$ *(11)* as desired. Since (1.36) is a rewriting of (1.35), it is immediate that *(12)* $\Leftrightarrow$ *(13)*. In summary, so far we have shown that *(1)*, *(2)*, *(7)*, *(9)*, *(10)*, *(11)*, *(12)*, and *(13)* are equivalent.

The implication *(6)* $\Rightarrow$ *(4)* is trivial and, based on (2.145), we have that *(4)* $\Rightarrow$ *(2)*. We focus next on *(1)* $\Rightarrow$ *(6)*. Suppose *(1)* holds and fix $f\in L^\infty(E,\sigma)$, $x\in E$, and $r\in(0,\infty)$ arbitrary. Then, using the notation $B_{cr}:=B_{\rho_\#}(x,cr)$ for $c>0$, we may write

$$\begin{aligned}
\int_{B_r\setminus E}|\Theta f|^2\delta_E^{2\upsilon-(m-d)}\,d\mu\;&\leq\;\int_{B_r\setminus E}|\Theta(f\mathbf{1}_{E\cap B_{2rC_\rho}})|^2\delta_E^{2\upsilon-(m-d)}\,d\mu\\
&\quad+\int_{B_r\setminus E}|\Theta(f\mathbf{1}_{E\setminus B_{2rC_\rho}})|^2\delta_E^{2\upsilon-(m-d)}\,d\mu=:I+II.
\end{aligned}\tag{7.2}$$

To estimate I we apply (1.26) and the property of E being d-dimensional ADR to obtain

$$I\leq C\int_{E\cap B_{2rC_\rho}}|f|^2\,d\sigma\leq C\|f\|^2_{L^\infty(E,\sigma)}\sigma(E\cap B_r).\tag{7.3}$$

As regards II, we first note that if $z\in B_r\setminus E$ and $y\in E\setminus B_{2rC_\rho}$ are arbitrary points then $\rho_\#(x,y)\leq C_\rho(\rho_\#(x,z)+\rho_\#(z,y))<C_\rho r+C_\rho\rho_\#(z,y)\leq\frac{1}{2}\rho_\#(x,y)+C_\rho\rho_\#(z,y)$ which implies $\rho_\#(z,y)\geq\rho_\#(x,y)/(2C_\rho)$. This, (1.23), and (3.20) then yield

$$\begin{aligned}
|\Theta(f\mathbf{1}_{E\setminus B_{2rC_\rho}})(z)|\;&\leq\;C\|f\|_{L^\infty(E,\sigma)}\int_{E\setminus B_{2rC_\rho}}\frac{1}{\rho_\#(x,y)^{d+\upsilon}}\,d\sigma(y)\\
&\leq\;C\|f\|_{L^\infty(E,\sigma)}\,r^{-\upsilon},\qquad\forall\,z\in B_r\setminus E.
\end{aligned}\tag{7.4}$$

Using this last estimate in II and applying (3.22) (with $R := r$ and $\gamma := m-d-2\upsilon$) we obtain

$$\begin{aligned} II &\le C\|f\|^2_{L^\infty(E,\sigma)} r^{-2\upsilon} \int_{B_r\setminus E} \delta_E^{2\upsilon-(m-d)}\, d\mu \\ &\le C\|f\|^2_{L^\infty(E,\sigma)} r^{-2\upsilon} r^{d+2\upsilon} \le C\|f\|^2_{L^\infty(E,\sigma)} \sigma(E\cap B_r). \end{aligned} \tag{7.5}$$

At this point, (1.31) follows from (7.2), (7.3), and (7.5), completing the proof of *(1)* $\Rightarrow$ *(6)*. Based on (2.145) we have that *(6)* $\Rightarrow$ *(3)* while *(3)* $\Rightarrow$ *(2)* is trivial.

Next, we shall show that *(8)* $\Rightarrow$ *(7)*. To this end, suppose *(8)* holds and let $\varepsilon_o := \min\{\varepsilon, a_0\}$, where ε is as in Lemma 2.23 and a_0 as in (2.50). Fix an arbitrary $Q \in \mathbb{D}(E)$ and define $\Delta_Q := B_{\rho_\#}\Big(x_Q, \frac{\varepsilon_o \ell(Q)}{2C_\rho}\Big) \cap E$. Then (2.50), (2.147) and the fact that E is d-dimensional ADR imply

$$\Delta_Q \subseteq Q, \quad B_{\rho_\#}\big(x_Q, \varepsilon_o\ell(Q)\big) \setminus E \subseteq T_E(Q), \quad \sigma(\Delta_Q) \approx \sigma(Q) = C\ell(Q)^d. \tag{7.6}$$

Hence, if we now define $b_Q := b_{\Delta_Q}$, where b_{Δ_Q} is the function associated to Δ_Q as in *(8)*, then b_{Δ_Q} satisfies (1.33) which, when combined with the support condition of b_{Δ_Q} and the last condition in (7.6), implies that b_Q satisfies the first two conditions in (1.32) (with $\widetilde{Q} = Q$). In order to show that b_Q also verifies the last condition in (1.32), we write

$$\begin{aligned} &\int_{T_E(Q)} |(\Theta\, b_Q)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\qquad = \int_{T_E(Q)\setminus B_{\rho_\#}\left(x_Q, \varepsilon_o\ell(Q)\right)} |(\Theta\, b_Q)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\qquad\quad + \int_{B_{\rho_\#}\left(x_Q, \varepsilon_o\ell(Q)\right)} |(\Theta\, b_Q)(x)|^2 \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) =: I_1 + I_2. \end{aligned} \tag{7.7}$$

To further estimate I_2, observe that if $x \in T_E(Q) \setminus B_{\rho_\#}\big(x_Q, \varepsilon_o\ell(Q)\big)$ and $y \in \Delta_Q$, then $\rho_\#(x,y) \ge \frac{\varepsilon_o}{2C_\rho}\ell(Q)$. This, (1.23), the first estimate in (1.32), and the last condition in (7.6), imply $|(\Theta\, b_Q)(x)| \le C\ell(Q)^{-\upsilon}$ for every $x \in T_E(Q) \setminus B_{\rho_\#}\big(x_Q, \varepsilon_o\ell(Q)\big)$. Hence, if we also recall (2.145) we have

$$\begin{aligned} I_1 &\le C\ell(Q)^{-2\upsilon} \int_{T_E(Q)\setminus B_{\rho_\#}\left(x_Q, \varepsilon_o\ell(Q)\right)} \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\le C\ell(Q)^{-2\upsilon} \int_{B_{\rho_\#}\left(x_Q, C\ell(Q)\right)\setminus E} \delta_E(x)^{2\upsilon-(m-d)}\, d\mu(x) \\ &\le C\ell(Q)^{-2\upsilon}\ell(Q)^{d+2\upsilon} \le C\sigma(Q), \end{aligned} \tag{7.8}$$

where the third inequality in (7.8) is a consequence of (3.22) (applied with $R = r = \ell(Q)$ and $\gamma := m-d-2\upsilon$). As for I_2, by recalling (1.33) and the last condition in (7.6), it is immediate that $I_2 \le C_0\sigma(\Delta_Q) \le C\sigma(Q)$. This, (7.7) and (7.8) show that b_Q also satisfies the last condition in (1.32) since the constants in our estimates are finite positive geometric, and independent of the choice of Q. This completes the proof of *(8)* $\Rightarrow$ *(7)*.

It is not difficult to see that *(1)* $\Rightarrow$ *(8)*. Indeed, by taking $b_\Delta := \mathbf{1}_\Delta$ for each surface ball Δ, the first two estimates in (1.33) are immediate while the third one is a consequence of (1.26) written for $f := b_\Delta$.

Trivially, *(2)* $\Rightarrow$ *(5)*. If we assume that *(5)* holds and for each $Q \in \mathbb{D}(E)$ we set $b_Q := b\mathbf{1}_Q$, then it is easy to verify based on (1.30) and the fact that b is para-accretive that (1.32) is satisfied by the family $\{b_Q\}_{Q\in\mathbb{D}(E)}$. Hence, *(5)* $\Rightarrow$*(7)*. The proof of Theorem 1.1 is therefore complete. $\square$

In the last part of this section we present the

Proof of Theorem 1.2. The idea is to apply Theorem 6.18 in the setting $\mathscr{X} := E \times [0,\infty)$ and $E \equiv E \times \{0\}$ (i.e., we identify $(y,0) \equiv y$ for every $y \in E$). Moreover, we let

$$\rho((x,t),(y,s)) := \max\{|x-y|,|t-s|\} \quad \text{for every} \quad (x,t),(y,s) \in E \times [0,\infty), \tag{7.9}$$

$$\mu := \sigma \otimes \mathcal{L}^1, \tag{7.10}$$

where $\mathcal{L}^1$ is the one-dimensional Lebesgue measure on $[0,\infty)$, and consider the integral kernel

$$\begin{aligned} &\theta : (\mathscr{X} \setminus E) \times E \to \mathbb{R} \\ &\theta\big((x,t),y\big) := 2^{-k}\psi_k(x-y) \;\text{ if }\; x,y \in E, t > 0 \;\text{ and }\; k \in \mathbb{Z},\; 2^k \le t < 2^{k+1}. \end{aligned} \tag{7.11}$$

Also, we let Θ be the integral operator defined in (3.4) corresponding to this choice of θ. Then it is not difficult to verify that $(\mathscr{X},\rho,\mu)$ is a $(d+1)$-ADR space, that $\alpha_\rho = 1$, that θ satisfies (3.1)-(3.3) for $a := 0$, $\alpha := 1$, $\upsilon := 1$, and that $\delta_E(x,t) = t$ for every $x \in E$ and $t \in [0,\infty)$. In particular, γ defined in (6.209) now equals 1. Fix some $\kappa > 0$ and observe that

$$\Gamma_\kappa(x) = \big\{(y,t) \in E \times (0,\infty) : \, |x-y| < (1+\kappa)t\big\}, \quad \forall\, x \in E. \tag{7.12}$$

In this context, for $f \in L^2(E,\sigma)$, we consider the square of the term in the left hand-side of (6.211) corresponding to $p = q = 2$ and use Fubini's Theorem, the property that E is d-ADR, and (7.11) to write

$$\begin{aligned} \Big\|\Big(\int_{\Gamma_\kappa(x)} |(\Theta f)(y,t)|^2 \frac{d\mu(y,t)}{\delta_E(y,t)^{d-1}}\Big)^{\frac12}\Big\|^2_{L^2_x(E,\sigma)} &= \int_E \int_{\Gamma_\kappa(x)} |(\Theta f)(y,t)|^2\, t^{1-d} d\mu(y,t)\, d\sigma(x) \\ &= \int_{E\times(0,\infty)} |(\Theta f)(y,t)|^2\, t^{1-d} \sigma\big(\{x \in E : \, y \in \Gamma_\kappa(x)\}\big)\, d\mu(y,t) \\ &= \int_{E\times(0,\infty)} |(\Theta f)(y,t)|^2\, t^{1-d} \sigma\big(E \cap B(y,(1+\kappa)t)\big)\, d\mu(y,t) \\ &\approx C \int_0^\infty \int_E |(\Theta f)(y,t)|^2\, t\, d\sigma(y)\, dt \\ &= C \sum_{k=-\infty}^{+\infty} \int_{2^k}^{2^{k+1}} \int_E \Big|\int_E 2^{-k}\psi_k(y-z) f(z)\, d\sigma(z)\Big|^2 d\sigma(y)\, t\, dt \\ &= C \sum_{k=-\infty}^{+\infty} \int_E \Big|\int_E \psi_k(y-z) f(z)\, d\sigma(z)\Big|^2 d\sigma(y). \end{aligned} \tag{7.13}$$

However, under the current assumptions on E, it was proved in [25, Theorem, p. 10] that there exists $C \in (0,\infty)$ with the property that

$$\sum_{k=-\infty}^{+\infty} \int_E \Big|\int_E \psi_k(x-y) f(y)\, d\sigma(y)\Big|^2 d\sigma(x) \le C \int_E |f|^2\, d\sigma, \quad \forall\, f \in L^2(E,\sigma). \tag{7.14}$$

Hence, we may apply Theorem 6.18 to conclude that there exists $C \in (0,\infty)$ such that estimate (6.210) is valid for every $q \in (1,\infty)$ and every $p \in \big(\frac{d}{d+1},\infty\big)$. In turn, reasoning as in (7.13), estimate (6.210) may be rewritten in the form

$$\Big\|\Big(\sum_{k=-\infty}^{+\infty}\fint_{y\in\Delta(x,(1+\kappa)2^k)}\Big|\int_E \psi_k(z-y)f(z)\,d\sigma(z)\Big|^q\,d\sigma(y)\Big)^{1/q}\Big\|_{L^p_x(E,\sigma)} \leq C'\|f\|_{H^p(E,\sigma)} \tag{7.15}$$

for every $f \in H^p(E,\sigma)$ and some $C' \in (0,\infty)$ independent of f. The desired conclusion now follows by observing that if $q \in (1,\infty)$ and $p \in \big(\frac{d}{d+1},\infty\big)$ are fixed, then there exists some $C \in (0,\infty)$ such that (1.38) holds for every $f \in H^p(E,\sigma)$ if and only if there exist $\kappa, C' \in (0,\infty)$ such that estimate (7.15) holds for every $f \in H^p(E,\sigma)$. Indeed, one direction is obvious, while the opposite one may be handled by observing that if $\psi \in C_0^\infty(\mathbb{R}^{n+1})$ is odd then $\widetilde{\psi}(x) := \psi(x/2^N)$, for some fixed sufficiently large $N \in \mathbb{N}$, is also odd, smooth and compactly supported, and satisfies $\widetilde{\psi}_k = 2^{dN}\psi_{k+N}$ for every $k \in \mathbb{Z}$. Writing (7.15) for $\widetilde{\psi}$ in place of ψ and shifting the index of summation in the left-hand side, the desired conclusion follows. $\square$

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